MAT 167: Advanced Linear Algebra
Solutions to SVD Exercises in HW8

Exercise 4.1 I already described the strategy for computing SVD of these small matrices via eigenvalue/eigenvector computation of A^TA.

(a) $\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$. First compute A^TA, which is $\begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$. The characteristic equation is:

$$\det(A^TA - \lambda I) = \det\left(\begin{bmatrix} 9-\lambda & 0 \\ 0 & 4-\lambda \end{bmatrix}\right) = (9-\lambda)(4-\lambda) = 0.$$

Thus, the eigenvalues are: $\lambda = 9, 4$. Therefore, we immediately get the singular values of A as $\sigma_1 = 3$, and $\sigma_2 = 2$. Now, we need to compute the eigenvectors. To do so, need to solve:

$$A^TA\boldsymbol{v}_j = \lambda_j \boldsymbol{v}_j, \quad \text{or} \quad (A^TA - \lambda_j I)\boldsymbol{v}_j = \boldsymbol{0}, \quad j = 1, 2.$$

Let $\boldsymbol{v}_1 = \begin{bmatrix} x & y \end{bmatrix}^T$.

$$\begin{bmatrix} 9-9 & 0 \\ 0 & 4-9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -5y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore $y = 0$ is a must. x can be chosen arbitrary if this is just for eigenvector computation, but we have additional constraint. We are computing the column vectors of the unitary (or orthogonal) matrix V of the SVD. Therefore we must choose $x = 1$ to make $\|\boldsymbol{v}_1\| = 1$. Thus $\boldsymbol{v}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$. Now for $\boldsymbol{v}_2$, we solve:

$$\begin{bmatrix} 9-4 & 0 \\ 0 & 4-4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, by the similar consideration, we get $\boldsymbol{v}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$. Therefore we computed so far $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, and $V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. To compute U, we use the relationship between $\boldsymbol{u}_j$ and $\boldsymbol{v}_j$, i.e., $A\boldsymbol{v}_j = \sigma_j \boldsymbol{u}_j$. Thus,

$$\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 3\boldsymbol{u}_1 \Longrightarrow \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2\boldsymbol{u}_2 \Longrightarrow \boldsymbol{u}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Therefore, $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Finally, we get:

$$A = U\Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. First compute $A^T A$, which is $\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$. The characteristic equation is:

$$\det(A^T A - \lambda I) = \det\left(\begin{bmatrix} 4-\lambda & 0 \\ 0 & 9-\lambda \end{bmatrix}\right) = (4-\lambda)(9-\lambda) = 0.$$

Thus, the eigenvalues are: $\lambda = 4, 9$. Therefore, we immediately get the singular values of A as $\sigma_1 = 3$, and $\sigma_2 = 2$. (We must have $\sigma_1 \geq \sigma_2$.) Now, for each eigenvalue, we compute the corresponding eigenvector.

$$\begin{bmatrix} 4-9 & 0 \\ 0 & 9-9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

By the same reasoning as in (a), we have $\boldsymbol{v}_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$. Now for $\boldsymbol{v}_2$, we solve:

$$\begin{bmatrix} 4-4 & 0 \\ 0 & 9-4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 5y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus we get $\boldsymbol{v}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$. Therefore we computed so far $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, and $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. To compute U, we again use the relationship between $\boldsymbol{u}_j$ and $\boldsymbol{v}_j$, i.e., $A\boldsymbol{v}_j = \sigma_j \boldsymbol{u}_j$. Thus,

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3\boldsymbol{u}_1 \Longrightarrow \boldsymbol{u}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\boldsymbol{u}_2 \Longrightarrow \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Therefore, $U = \begin{bmatrix} 0 & 11 & 0 \end{bmatrix}$. Finally, we get:

$$A = U\Sigma V^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(c) $\begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. First compute $A^T A$, which is $\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$. The characteristic equation is:

$$\det(A^T A - \lambda I) = \det\left(\begin{bmatrix} 0-\lambda & 0 \\ 0 & 4-\lambda \end{bmatrix}\right) = -\lambda(4-\lambda) = 0.$$

Thus, the eigenvalues are: $\lambda = 4, 0$. Therefore, we immediately get the singular values of A as $\sigma_1 = 2$, and $\sigma_2 = 0$ (We must choose $\sigma_1 \geq \sigma_2$). Now, we compute the eigenvectors as before.

$$\begin{bmatrix} 0-4 & 0 \\ 0 & 4-4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus $\boldsymbol{v}_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$. Now for $\boldsymbol{v}_2$, we solve:

$$\begin{bmatrix} 0-0 & 0 \\ 0 & 4-0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 4y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, we get $\boldsymbol{v}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$. Therefore we computed so far $\hat{\Sigma} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, and $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. To compute $\hat{U}$, we use the relationship between $\boldsymbol{u}_j$ and $\boldsymbol{v}_j$, i.e., $A\boldsymbol{v}_j = \sigma_j \boldsymbol{u}_j$. Thus,

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2\boldsymbol{u}_1 \Longrightarrow \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Now, to determine $\boldsymbol{u}_2$, the singular value $\sigma_2 = 0$, and

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0\boldsymbol{u}_2.$$

This equation is satisfied by any $\boldsymbol{u}_2 \in \mathbb{R}^3$. Now, however, we must choose $\boldsymbol{u}_2$ that is perpendicular to $\boldsymbol{u}_1$ and the unit length. Thus, we choose $\boldsymbol{u}_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$. Therefore, $\hat{U} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Finally, we get the following *reduced* SVD:

$$A = \hat{U}\hat{\Sigma}V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

If we want to compute the *full* SVD, we need to compute $\boldsymbol{u}_3$, which must be of unit norm and perpendicular to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. Thus, we can easily obtain $\boldsymbol{u}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. The singular value σ_3 must be 0 since all the singular values must be nonnegative and ordered in a non-increasing manner. Thus, we have:

$$A = U\Sigma V^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(d) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. First compute A^TA, which is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The characteristic equation is:

$$\det(A^TA - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2 - 1^2 = \lambda(2-\lambda) = 0.$$

Thus, the eigenvalues are: $\lambda = 2, 0$. Therefore, we immediately get the singular values of A as $\sigma_1 = \sqrt{2}$, and $\sigma_2 = 0$. (We must have $\sigma_1 \geq \sigma_2$.) Now, for each eigenvalue, we compute the corresponding eigenvector.

$$\begin{bmatrix} 1-2 & 1 \\ 1 & 1-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x+y \\ x-y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus we have the solution $\boldsymbol{v}_1 = \begin{bmatrix} x & x \end{bmatrix}^T$ for arbitrary $x \in \mathbb{R}$ in general, but we must have $\|\boldsymbol{v}_1\| = 1$. Thus we have $\boldsymbol{v}_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$. Now for $\boldsymbol{v}_2$, we solve:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x+y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus we get $\boldsymbol{v}_2 = \begin{bmatrix} x & -x \end{bmatrix}^T$. Again to have the unit norm, we get: $\boldsymbol{v}_2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T$. Therefore we computed so far $\Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$, and $V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$. To compute U, we again use the relationship between $\boldsymbol{u}_j$ and $\boldsymbol{v}_j$, i.e., $A\boldsymbol{v}_j = \sigma_j \boldsymbol{u}_j$. Thus,

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \sqrt{2}\boldsymbol{u}_1 \Longrightarrow \boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

For $\boldsymbol{u}_2$, because $\sigma_2 = 0$, we can choose the unit vector perpendicular to $\boldsymbol{u}_1$. Thus, we get $\boldsymbol{u}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$. Therefore, $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Finally, we get:

$$A = U\Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

(e) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. First compute A^TA, which is $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$. The characteristic equation is:

$$\det(A^TA - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix}\right) = (2-\lambda)^2 - 2^2 = \lambda(4-\lambda) = 0.$$

Thus, the eigenvalues are: $\lambda = 4, 0$. Therefore, we immediately get the singular values of A as $\sigma_1 = 2$, and $\sigma_2 = 0$. (We must have $\sigma_1 \geq \sigma_2$.) Now, for each eigenvalue, we compute the corresponding eigenvector.

$$\begin{bmatrix} 2-4 & 2 \\ 2 & 2-4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2x+2y \\ 2x-2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus we have the solution $\boldsymbol{v}_1 = \begin{bmatrix} x & x \end{bmatrix}^T$ for arbitrary $x \in \mathbb{R}$ in general, but we must have $\|bv_1\| = 1$. Thus we have $\boldsymbol{v}_1 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T$. Now for $\boldsymbol{v}_2$, we solve:

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + 2y \\ 2x + 2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus we get $\boldsymbol{v}_2 = \begin{bmatrix} x & -x \end{bmatrix}^T$. Again to have the unit norm, we get: $\boldsymbol{v}_2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T$. Therefore we computed so far $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, and $V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$. To compute U, we again use the relationship between $\boldsymbol{u}_j$ and $\boldsymbol{v}_j$, i.e., $A\boldsymbol{v}_j = \sigma_j \boldsymbol{u}_j$. Thus,

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = 2\boldsymbol{u}_1 \Longrightarrow \boldsymbol{u}_1 = \begin{bmatrix} 1/\sqrt{2} 1/\sqrt{2} \end{bmatrix}.$$

For $\boldsymbol{u}_2$, because $\sigma_2 = 0$, we can choose the unit vector perpendicular to $\boldsymbol{u}_1$. Thus, we can get, for example, $\boldsymbol{u}_2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T$. Therefore, $U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$. Finally, we get:

$$A = U\Sigma V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

Exercise 4.4 This is *false*. The best explanation is to consider the following counterexample. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. It is easy to see that the singular values of A and B are the same, i.e., $\sigma_1 = 1$, $\sigma_2 = 0$. (Do the similar computation as in Exercise 4.1.) The question is, therefore, where there is any unitary matrix Q that satisfies $A = QBQ^*$, in other words, $AQ = QB$. OK, so let's $Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then,

$$AQ = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = QB = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}.$$

Thus we must have $a = b = c = 0$. Now, $Q = \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$, which cannot be unitary, i.e., $Q^*Q = I$, regardless of the values of d. Thus the statement of this exercise is false.

Exercise 4.5 The easiest way to prove this statement is to use the following fact:

Theorem: Let $B \in \mathbb{R}^{n\times n}$. Then, B is a real symmetric matrix if and only if B has an real eigenvalue-eigenvector decomposition, i.e., $B = P\Lambda P^T$, where P is an $n \times n$ orthogonal matrix and Λ is an $n \times n$ real-valued diagonal matrix.

If we decide to use the above theorem, it is easy to prove this exercise. For any $A \in \mathbb{R}^{m\times n}$, it has a SVD, $A = U\Sigma V^*$. At this point, we do not know whether U and V are real-valued matrix. The only thing we know is that they are unitary. Now, A^TA is an $n \times n$ real symmetric matrix. Thus, A^TA must have real-valued eigenvalues and eigenvectors. Because the eigenvalues of A^TA is the square of the singular values of A, we have $A^TA = V\Sigma^T\Sigma V^*$ where V plays the role of P, and $\Sigma^T\Sigma$ plays the role of Λ in the above theorem. Therefore, V must be a real-valued orthogonal matrix. Now, consider AA^T and again this is a real symmetric matrix. With the same reasoning, U must be a real-valued orthogonal matrix.

Another possible approach to prove this problem is to follow the proof of Theorem 4.1 in the Lecture 4 note by replace every occurrence of $\mathbb{C}$ and "unitary" to $\mathbb{R}$ and "orthogonal".

Exercise 5.1 In this problem, as I announced in my email, the matrix A is $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$. We just follow the same strategy as Exercise 4.1. Since $A^T A = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}$, the characteristic equation is:

$$\det(A^T A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 2 & 8-\lambda \end{bmatrix}\right) = (1-\lambda)(8-\lambda) - 2^2 = \lambda^2 - 9\lambda + 4 = 0.$$

Thus $\lambda = \frac{9 \pm \sqrt{65}}{2}$. Now, the singular values are: $\sigma_1 = \sqrt{\frac{9+\sqrt{65}}{2}} \approx 2.9208$, and $\sigma_2 = \sqrt{\frac{9-\sqrt{65}}{2}} \approx 0.6847$.

Just in case, the eigenvectors of $A^T A$ are:

$$\begin{bmatrix} 1 - \frac{9+\sqrt{65}}{2} & 2 \\ 2 & 8 - \frac{9+\sqrt{65}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{7+\sqrt{65}}{2}x + 2y \\ 2x + \frac{7-\sqrt{65}}{2}y \end{bmatrix} = \mathbf{0}.$$

From here we get $\frac{7+\sqrt{65}}{2}x = 2y$. Then, setting, say $x = 4$, $y = 7 + \sqrt{65}$. But $\boldsymbol{v}_1$ must be of the unit norm.Thus, by normalizing it to get:

$$\boldsymbol{v}_1 = \begin{bmatrix} \sqrt{\frac{8}{65+7\sqrt{65}}} & \sqrt{\frac{57+7\sqrt{65}}{65+7\sqrt{65}}} \end{bmatrix}^T.$$

Similarly, for $\boldsymbol{v}_2$, we have:

$$\begin{bmatrix} 1 - \frac{9-\sqrt{65}}{2} & 2 \\ 2 & 8 - \frac{9-\sqrt{65}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{7-\sqrt{65}}{2}x + 2y \\ 2x + \frac{7+\sqrt{65}}{2}y \end{bmatrix} = \mathbf{0}.$$

From here we get $\frac{7-\sqrt{65}}{2}x = 2y$. setting, say $x = -4$, $y = -7 + \sqrt{65}$. But $\boldsymbol{v}_1$ must be of the unit norm.Thus, by normalizing it to get:

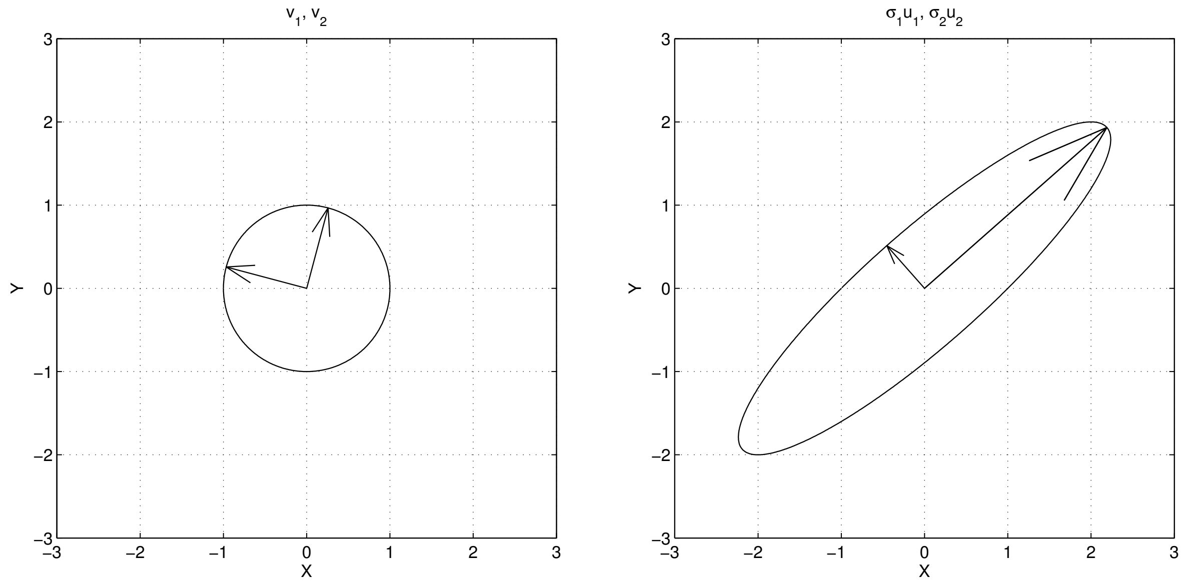
$$\boldsymbol{v}_2 = \begin{bmatrix} -\sqrt{\frac{8}{65-7\sqrt{65}}} & \sqrt{\frac{57-7\sqrt{65}}{65-7\sqrt{65}}} \end{bmatrix}^T.$$

Now, we can compute U via:

$$\boldsymbol{u}_1 = \frac{1}{\sigma_1} A \boldsymbol{v}_1 = \sqrt{\frac{2}{9+\sqrt{65}}} \begin{bmatrix} \frac{\sqrt{8}+2\sqrt{57+7\sqrt{65}}}{\sqrt{65+7\sqrt{65}}} \\ \frac{2\sqrt{57+7\sqrt{65}}}{\sqrt{65+7\sqrt{65}}} \end{bmatrix} = \frac{\sqrt{2}}{2\sqrt{65+8\sqrt{65}}} \begin{bmatrix} \sqrt{2} + \sqrt{57+7\sqrt{65}} \\ \sqrt{57+7\sqrt{65}} \end{bmatrix}.$$

$$\boldsymbol{u}_2 = \frac{1}{\sigma_2} A \boldsymbol{v}_2 = \sqrt{\frac{2}{9-\sqrt{65}}} \begin{bmatrix} \frac{-\sqrt{8}+2\sqrt{57-7\sqrt{65}}}{\sqrt{65+7\sqrt{65}}} \\ \frac{2\sqrt{57-7\sqrt{65}}}{\sqrt{65-7\sqrt{65}}} \end{bmatrix} = \frac{\sqrt{2}}{2\sqrt{65-8\sqrt{65}}} \begin{bmatrix} -\sqrt{2} + \sqrt{57-7\sqrt{65}} \\ \sqrt{57-7\sqrt{65}} \end{bmatrix}.$$

The following figure shows how the unit sphere (circle) in $\mathbb{R}^2$ is mapped to the ellipse by this matrix A.



(a) Unit Circle S (b) AS

Exercise 5.3

$$A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}$$

(a) As before, we first compute the eigenvalues and eigenvectors of $A^T A$.

$$A^T A = \begin{bmatrix} -2 & -10 \\ 11 & 5 \end{bmatrix} \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} = \begin{bmatrix} 104 & -72 \\ -72 & 146 \end{bmatrix}.$$

Thus, the characteristic equation is:

$$\det(A^T A - \lambda I) = \det \begin{bmatrix} 104 - \lambda & -72 \\ -72 & 146 - \lambda \end{bmatrix} = \lambda^2 - 250\lambda + 10000 = 0,$$

$$\Longrightarrow (\lambda - 200)(\lambda - 50) = 0 \Longrightarrow \lambda = 200, 50 \Longrightarrow \sigma_1 = 10\sqrt{2}, \sigma_2 = 5\sqrt{2}.$$

Now let's compute the eigenvectors. For the first eigenvector $\boldsymbol{v}_1 = \begin{bmatrix} x & y \end{bmatrix}^T$, we need to solve $A^T A \boldsymbol{v}_1 = 200 \boldsymbol{v}_1$:

$$\begin{bmatrix} 104x - 72y \\ -72x + 146y \end{bmatrix} = \begin{bmatrix} 200x \\ 200y \end{bmatrix}$$

Thus,

$$\begin{aligned} 96x &= -72y \\ -72x &= 54y, \end{aligned}$$

which are the same equation: $4x = -3y$. Therefore, we can take, say, $x = -3; y = 4$. Note that in this case, it is impossible for x and y to have the same sign. Since $\|\boldsymbol{v}_1\| = 1$, we normalize it to have $\boldsymbol{v}_1 = \begin{bmatrix} -3/5 & 4/5 \end{bmatrix}$. (Of course, it is also possible to have $\boldsymbol{v}_1 = \begin{bmatrix} 3/5 & -4/5 \end{bmatrix}$. This choice is up to you. Both are correct.) As for the second eigenvector $\boldsymbol{v}_2$, we have to solve:

$$\begin{bmatrix} 104x - 72y \\ -72x + 146y \end{bmatrix} = \begin{bmatrix} 50x \\ 50y \end{bmatrix}$$

Thus,

$$\begin{aligned} 54x &= 72y \\ 3x &= 4y, \end{aligned}$$

which are the same equation: $3x = 4y$. In this case, we can take say, $x = 4; y = 3$, of the same sign. After the normalization, we have $\boldsymbol{v}_2 = \begin{bmatrix} 4/5 & 3/5 \end{bmatrix}$.

So far, we computed:

$$\Sigma = \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix},$$

$$V = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

As for U, we use the formula $\boldsymbol{u}_j = \frac{1}{\sigma_j} A \boldsymbol{v}_j$, $j = 1, 2$:

$$\boldsymbol{u}_1 = \frac{1}{10\sqrt{2}} \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

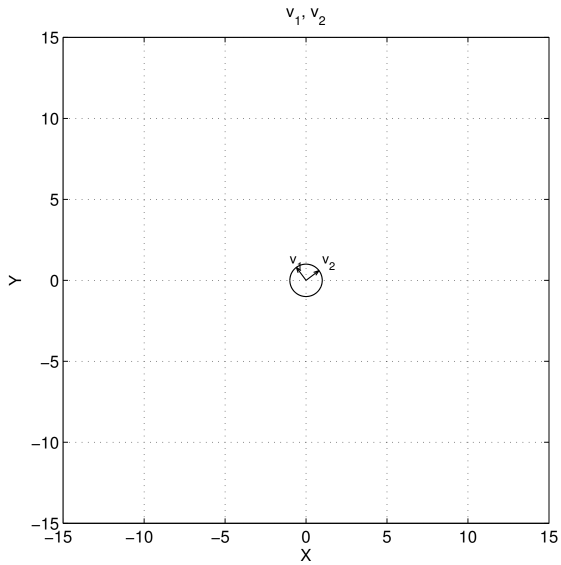
$$\boldsymbol{u}_2 = \frac{1}{5\sqrt{2}} \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}.$$

Therefore we have:

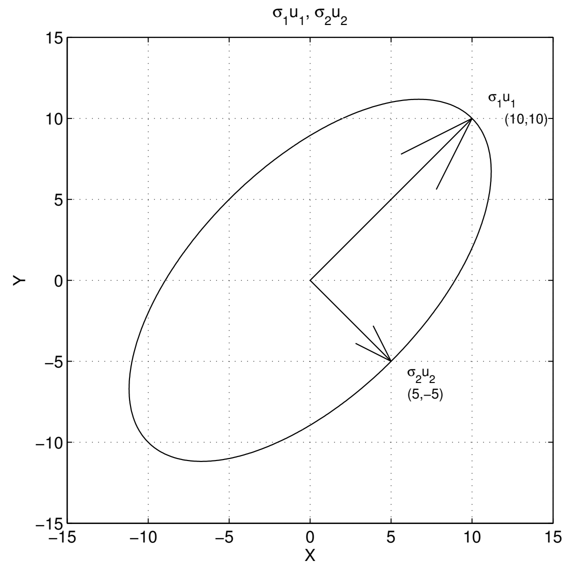
$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Finally, we have the following SVD of A:

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix} \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$



(b.1) Unit Circle S



(b.2) AS

(b) See the above figures.

(c) The 1-norm of an matrix A is the largest 1-norm of the column vectors of A. Thus,

$$\|A\|_1 = \max_{j=1,2} \|A(:,j)\|_1 = 16.$$

The 2-norm of A is of course the largest singular value σ_1. Thus,

$$\|A\|_2 = \sigma_1 = 10\sqrt{2}.$$

The ∞-norm of A is the largest 1-norm of the *row* vectors of A. Thus,

$$\|A\|_\infty = \max_{i=1,2} \|A(i,:)\|_1 = 15.$$

(d) In this case, the diagonal elements of Σ are not zeros, so we can compute the exact inverse of A:

$$\begin{aligned} A^{-1} = V\Sigma^{-1}U^* &= \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{10\sqrt{2}} & 0 \\ 0 & \frac{1}{5\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{10} & 0 \\ 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \frac{1}{100} \begin{bmatrix} 5 & -11 \\ 10 & -2 \end{bmatrix}. \end{aligned}$$

(e)

$$\det(A - \lambda I) = \det \begin{bmatrix} -2-\lambda & 11 \\ -10 & 5-\lambda \end{bmatrix} = (-2-\lambda)(5-\lambda) + 100 = 0$$

$$\Longrightarrow (\lambda - 5)(\lambda + 2) + 100 = 0 \Longrightarrow \lambda^2 - 3\lambda + 100 = 0.$$

This leads to

$$\lambda = \frac{3 \pm \sqrt{391}\mathrm{i}}{2},$$

i.e.,

$$\lambda_1 = \frac{3 + \sqrt{391}\mathrm{i}}{2}, \quad \lambda_2 = \frac{3 - \sqrt{391}\mathrm{i}}{2}.$$

(f) This is just a simple computation.

$$\det(A) = -2 * 5 + 110 = 100.$$

$$\lambda_1 \lambda_2 = \frac{9 + 391}{4} = 100.$$

$$|\det(A)| = \sigma_1 \sigma_2 = 10\sqrt{2} \cdot 5\sqrt{2} = 100.$$

(g) The area of the ellipsoid is π times the length of the major axis times the length of the minor axis. Thus, it is $\pi \sigma_1 \sigma_2 = 100\pi$.