MAT 167: Advanced Linear Algebra
Midterm Exam Solutions

Problem 1 (20 pts) We would like to find the best line for the given three points $(0, 1)$, $(1, 1)$, $(2, 2)$ in the plane $\mathbb{R}^2$ in the least squares sense.

(a) (6 pts) Let us write an equation of line as $y = px + q$. Then write a system of equation of the form $A \begin{bmatrix} q \\ p \end{bmatrix} = \boldsymbol{b}$, as if the line passes through all the three points.

Answer: Entering $x = 0, 1, 2$ and $y = 1, 1, 2$, respectively to the equation of line $y = px + q$ yields the following system of equations.

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} q \\ p \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}_{\boldsymbol{b}}$$

(b) (6 pts) Show that this system is inconsistent (i.e., therefore, the line cannot pass all the three points after all).

Answer: Form the augmented matrix, and obtain its Reduced Row Echelon Form:

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

So, the last element 1 does not match with the zero elements of the last row. Therefore, this system is inconsistent.

(c) (8 pts) Now, form the normal equation, and compute the best line by solving this normal equation.

Answer: Form the normal equation.

$$\begin{aligned} A^T A \boldsymbol{x} &= A^T \boldsymbol{b}, \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \\ \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} &= \begin{bmatrix} 4 \\ 5 \end{bmatrix}. \end{aligned}$$

Therefore,

$$\begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 5/6 \\ 1/2 \end{bmatrix}.$$

Thus the best line is $\underline{y = \frac{1}{2}x + \frac{5}{6}}$.

Problem 2 (25 pts) Consider the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

(a) (5 pts) Find a basis of $\mathcal{R}(A)$.

Answer: Do the Gaussian Elimination of A to get:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} = E_A$$

Thus it is clear that the first two columns are the basic columns of A, which are the basis of $\mathcal{R}(A)$. In other words,

$$\underline{\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.}$$

(b) (5 pts) What is the value of $\operatorname{rank}(A)$ and the dimension of $\mathcal{N}(A)$? Use the information obtained in Part (a).

Answer: From Part (a), we get $\operatorname{rank}(A) = 2$. Now, the number of columns of A is 3. So, the Rank-Nullity Theorem says:

$$\operatorname{rank}(A) + \dim(\mathcal{N}(A)) = 3.$$

Thus, $\dim(\mathcal{N}(A)) = 3 - 2 = \underline{1}$.

(c) (5 pts) Find a basis of $\mathcal{N}(A)$.

Answer: We need to solve the homogeneous system of equations, $A\boldsymbol{x} = \mathbf{0}$. Since we already know the RREF E_A, we need to solve:

$$E_A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From this, we get the following equations:

$$\begin{cases} x + 2z & = 0 \\ y - z & = 0. \end{cases}$$

Thus, z is a free variable, and the general solution is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2z \\ z \\ z \end{bmatrix} = z \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = z \cdot \boldsymbol{h}_1.$$

Therefore, the basis of $\mathcal{N}(A)$ is

$$\underline{\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}}.$$

(d) (5 pts) Find a basis of $\mathcal{R}(A^T)$.

Answer: The nonzero rows of E_A form a basis $\mathcal{R}(A^T)$. Therefore, the answer is:

$$\underline{\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}}.$$

(e) (5 pts) What is $\mathcal{N}(A^T)$ in this case?

Answer: Again using the Rank-Nullity Theorem for A^T, we have:

$$\dim(\mathcal{R}(A^T)) + \dim(\mathcal{N}(A^T)) = 2$$

And clearly $\dim(\mathcal{R}(A^T)) = \operatorname{rank}(A^T) = \operatorname{rank}(A) = 2$. Therefore, $\dim(\mathcal{N}(A^T)) = 0$, which means that $\underline{\mathcal{N}(A^T) = \{\mathbf{0}\}}$.

Problem 3 (25 pts) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transform defined by $T(x, y) = (2x + y, x + 2y)$.

Consider the vector $\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}.$$

Let $U = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ be the matrix representing this basis.

(a) (5 pts) Determine $[T]_{\mathcal{B}}$ and $[\boldsymbol{v}]_{\mathcal{B}}$.

Answer: Let A be the matrix associated with this linear transformation T. Then,

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ x + 2y \end{bmatrix} \Longrightarrow A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Now, we know that

$$[T]_{\mathcal{B}} = U^{-1} A U = \frac{1}{2} \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \underline{\frac{1}{2} \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}}.$$

As for $[\boldsymbol{v}]_{\mathcal{B}}$, it is easy to get:

$$[\boldsymbol{v}]_{\mathcal{B}} = U^{-1} \boldsymbol{v} = \frac{1}{2} \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \underline{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}.$$

(b) (5 pts) Compute $[T(\boldsymbol{v})]_{\mathcal{B}}$ and verify that $[T]_{\mathcal{B}} [\boldsymbol{v}]_{\mathcal{B}} = [T(\boldsymbol{v})]_{\mathcal{B}}$.

Answer: In this case, $T(\boldsymbol{v}) = A\boldsymbol{v}$. So,

$$[T(\boldsymbol{v})]_{\mathcal{B}} = [A\boldsymbol{v}]_{\mathcal{B}} = U^{-1} A \boldsymbol{v} = \frac{1}{2} \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

On the other hand, using the results of Part (a), we have:

$$[T]_{\mathcal{B}} [\boldsymbol{v}]_{\mathcal{B}} = \frac{1}{2} \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

So, they surely agree.

(c) (5 pts) Now, let a new basis in $\mathbb{R}^2$ be

$$\widetilde{\mathcal{B}} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

Let $V = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ be the matrix representing this basis. Now determine the change of basis matrix $[I]_{\mathcal{B}\widetilde{\mathcal{B}}}$.

Answer: The matrix we want to compute is:

$$[I]_{\mathcal{B}\widetilde{\mathcal{B}}} = V^{-1}U = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \underline{\frac{1}{2}\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}}.$$

(d) (5 pts) determine $[T]_{\widetilde{\mathcal{B}}}$.

Answer: This can be easily done as follows:

$$[T]_{\widetilde{\mathcal{B}}} = V^{-1}AV = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \underline{\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}}.$$

(e) (5 pts) determine $[T]_{\mathcal{B}\widetilde{\mathcal{B}}}$ and demonstrate that $[T]_{\widetilde{\mathcal{B}}}[I]_{\mathcal{B}\widetilde{\mathcal{B}}} = [T]_{\mathcal{B}\widetilde{\mathcal{B}}}$.

Answer: First we have:

$$[T]_{\mathcal{B}\widetilde{\mathcal{B}}} = V^{-1}AU = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \underline{\frac{1}{2}\begin{bmatrix} 3 & 9 \\ -1 & 1 \end{bmatrix}}.$$

On the other hand, we have:

$$[T]_{\widetilde{\mathcal{B}}}[I]_{\mathcal{B}\widetilde{\mathcal{B}}} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & 9 \\ -1 & 1 \end{bmatrix}.$$

Therefore, they completely agree.

Problem 4 (15 pts) Let $A, B \in \mathbb{R}^{m\times n}$, and let E_A, E_B be their reduced row echelon forms, respectively. Tell the following statements are true or false:

(a) (3 pts) If $A \overset{\text{row}}{\sim} B$, then automatically $A \sim B$.

Answer: <u>True</u>. Because the row equivalence is a special case of the equivalence.

(b) (3 pts) $A \overset{\text{col}}{\sim} B$ if and only if $E_A = E_B$.

Answer: <u>False</u>. $E_{A^T} = E_{B^T}$ is the necessary and sufficient condition for the column equivalence.

(c) (3 pts) $\dim(\mathcal{R}(A)) = \operatorname{rank}(A^T)$.

Answer: <u>True</u> since $\operatorname{rank}(A^T) = \operatorname{rank}(A)$.

(d) (3 pts) If $\operatorname{rank}(A) < n$, then the column vectors of A is linearly independent.

Answer: <u>False</u>. The columns of A are linearly dependent in this case.

(e) (3 pts) Let $m = n$ and suppose the LU factorization of $A = LU$ exists. Then, the best way to solve $A\boldsymbol{x} = \boldsymbol{b}$ in this case is to solve $L\boldsymbol{y} = \boldsymbol{b}$ for $\boldsymbol{y}$ via backward substitution followed by solving $U\boldsymbol{x} = \boldsymbol{y}$ for $\boldsymbol{x}$ via forward substitution.

Answer: <u>False</u>. $L\boldsymbol{y} = \boldsymbol{b}$ should be solved via *forward* substitution, and $U\boldsymbol{x} = \boldsymbol{y}$ should be solved via *backward* substitution.

Problem 5 (15 pts) Let $A \in \mathbb{R}^{m\times n}$.

(a) (7 pts) Show that $\mathcal{N}(A)$ is a subspace of $\mathbb{R}^n$.

Answer: First of all, by definition, $\mathcal{N}(A) \subset \mathbb{R}^n$. Now take any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{N}(A)$ and any $\alpha \in \mathbb{R}$. Then,

$$A(\boldsymbol{x}+\boldsymbol{y}) = A\boldsymbol{x} + A\boldsymbol{y} = \boldsymbol{0} + \boldsymbol{0} = \boldsymbol{0}.$$

So, $\boldsymbol{x}+\boldsymbol{y} \in \mathcal{N}(A)$. Finally,

$$A(\alpha\boldsymbol{x}) = \alpha A\boldsymbol{x} = \alpha \cdot \boldsymbol{0} = \boldsymbol{0}.$$

Thus, $\alpha\boldsymbol{x} \in \mathcal{N}(A)$. Therefore, $\mathcal{N}(A)$ is a subspace of $\mathbb{R}^n$.

(b) (8 pts) Consider now a system of nonhomogeneous equations $A\boldsymbol{x} = \boldsymbol{b}$, where obviously $\boldsymbol{x} \in \mathbb{R}^n$, and $\boldsymbol{b} \in \mathbb{R}^m$. Show that if $\mathcal{N}(A) = \{\boldsymbol{0}\}$ and $\boldsymbol{b} \in \mathcal{R}(A)$, then $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution. [Hint: Assume there are two distinct solutions, $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Then what happens?]

Answer: Since $\boldsymbol{b} \in \mathcal{R}(A)$, there exists at least one solution. Suppose now that there exist two distinct solutions $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$. Then, we have

$$\begin{aligned} A\boldsymbol{x}_1 &= \boldsymbol{b} \\ A\boldsymbol{x}_2 &= \boldsymbol{b}. \end{aligned}$$

Subtracting the second equation from the first gives us:

$$A\boldsymbol{x}_1 - A\boldsymbol{x}_2 = A(\boldsymbol{x}_1 - \boldsymbol{x}_2) = \boldsymbol{b} - \boldsymbol{b} = \boldsymbol{0}.$$

Therefore $\boldsymbol{x}_1 - \boldsymbol{x}_2 \in \mathcal{N}(A)$. But $\mathcal{N}(A) = \{\boldsymbol{0}\}$ by the assumption. Thus, we must have $\boldsymbol{x}_1 - \boldsymbol{x}_2 = \boldsymbol{0}$. This means that $\boldsymbol{x}_1 = \boldsymbol{x}_2$, which is a contradiction. Therefore, there exists only one solution.