

# Continuation of Vector/Matrix Review

Note Title

4/9/2012

## \* Range & Nullspace (or Kernel)

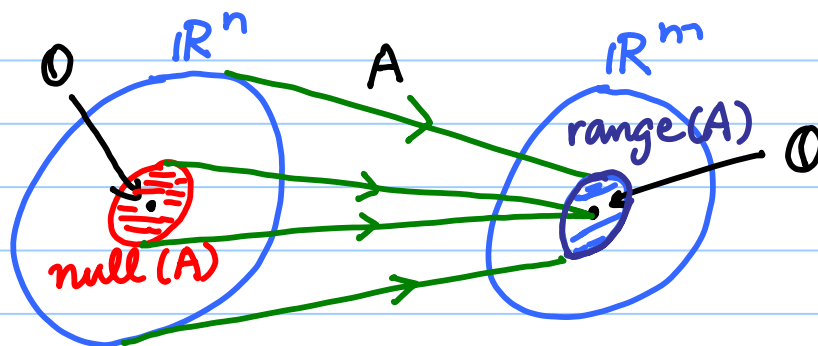
Def.  $A \in \mathbb{R}^{m \times n}$ .

$$\text{range}(A) := \{y \in \mathbb{R}^m \mid y = Ax, x \in \mathbb{R}^n\}$$

often written as  $\text{Ran}(A)$  or  $\text{Im}(A)$ .

This is also called the column space of  $A$ . image

$\text{null}(A) := \{x \in \mathbb{R}^n \mid Ax = 0\}$   
is called the nullspace (or kernel) of  $A$   
 $\text{Ker}(A)$



Thm  $\text{range}(A) = \text{span}\{a_1, \dots, a_n\}$   
= a set of all possible  
linear combi. of  $\{a_1, \dots, a_n\}$

(Proof) Need to show two things

(1)  $\text{range}(A) \subset \text{span}\{a_1, \dots, a_n\}$

(2)  $\text{span}\{a_1, \dots, a_n\} \subset \text{range}(A)$

Now, (1) is easy since any  $y \in \text{range}(A)$   
by definition,  $\exists x \in \mathbb{R}^n$  s.t.  $y = Ax$ .

This is a lin. combi. of col vectors of  $A$   
So,  $y \in \text{span}\{a_1, \dots, a_n\}$ .

(2) Take any  $y \in \text{span}\{a_1, \dots, a_n\}$ .  
 By definition,  $\exists \{x_1, \dots, x_n\}$  s.t.  
 $y = x_1 a_1 + \dots + x_n a_n = A X \in \text{range}(A)$   
 by setting  $X = (x_1, \dots, x_n)^T$  //

### ★ Linear Independence, Bases

Def. The vectors  $\{a_1, \dots, a_n\}$ ,  $a_j \in \mathbb{R}^m$   
 are called linearly independent if  

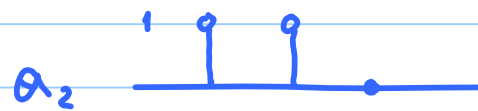
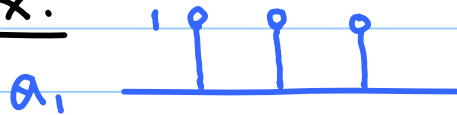
$$\sum_{j=1}^n x_j a_j = 0 \iff x_j = 0, 1 \leq j \leq n$$

A set of  $m$  linearly independent  
 vectors in  $\mathbb{R}^m$  is called a basis  
 in  $\mathbb{R}^m$ .  $\Rightarrow$  a matrix representation  
 of a basis in  $\mathbb{R}^m$  is an  $m \times m$  matrix.  
 Note that any vector in  $\mathbb{R}^m$  can be  
 written as a lin. combi. of the  $m$   
basis vectors in  $\mathbb{R}^m$

Def. The dimension of  $\text{span}\{a_1, \dots, a_n\}$   
 is the maximal number of linearly  
independent vectors among  $\{a_1, \dots, a_n\}$

i.e., if  $\exists j$ ,  $a_j = x_1 a_1 + \dots + x_{j-1} a_{j-1}$   
 $+ x_{j+1} a_{j+1} + \dots + x_n a_n$   
 then such  $a_j$  is useless (or redundant)

Ex.



In  $\mathbb{R}^3$ , these are linearly dependent.

$$a_1 = a_2 + a_3$$

So  $\dim \text{span}\{a_1, a_2, a_3\} = 2$ .

We cannot write any vector in  $\mathbb{R}^3$  by a lin. combi. of  $\{a_2, a_3\}$ .

Only a certain subset of vectors in  $\mathbb{R}^3$  can be written as a lin. combi of  $\{a_2, a_3\}$  (no control on the first and second entries of a vector in  $\mathbb{R}^3$ .)

## \* Rank

Def. The column rank of  $A$

$$:= \dim(\text{range}(A))$$

= # of linearly indep. col. vec's of  $A$ .

The row rank of  $A$

$$:= \dim(\text{range}(A^T))$$

= # of linearly indep. row vec's of  $A$ .

$$\underline{\text{rank}(A)} := \dim(\text{range}(A))$$

$A \in \mathbb{R}^{m \times n}$  is said to be of full rank if  $\text{rank}(A) = \min(m, n)$ .

Thm.  $A \in \mathbb{R}^{m \times n}$ ,  $m \geq n$  is of full rank  
 $\Leftrightarrow \forall x, y \in \mathbb{R}^n$ ,  $x \neq y$ ,  
 $Ax \neq Ay$ .

(Proof)  $[\Rightarrow]$  If  $\text{rank}(A) = n$ , i.e., full rank,  
then  $\{a_1, \dots, a_n\}$  are lin. indep.

So, they form a basis of  $\text{range}(A)$ .

This means that  $\forall b \in \text{range}(A)$ ,

There exists a unique ...

$\exists! x \in \mathbb{R}^n$  s.t.  $b = \sum_{j=1}^n x_j a_j$ .

$[\Leftarrow]$  Suppose  $A$  is not of full rank.

Then  $\{a_1, \dots, a_n\}$  are linearly dep.

i.e.,  $\exists c \in \mathbb{R}^n$ ,  $c \neq 0$  s.t.

$$\sum_{j=1}^n c_j a_j = 0, \text{ i.e., } Ac = 0.$$

Then set  $y = x + c \neq x$ .

But  $Ay = A(x + c) = Ax + Ac$

$= Ax$  contradiction! #

### ★ Inverse

Def.  $A$  is said to be nonsingular  
or invertible  $\Leftrightarrow A$  is square and  
of full rank

Hence  $A \in \mathbb{R}^{m \times m}$ : nonsingular

$\Rightarrow \{a_1, \dots, a_m\}$  form a basis of  $\mathbb{R}^m$

That means: the canonical basis vector  $e_j \in \mathbb{R}^m$  can also be written as a lin. combi. of  $\{a_1, \dots, a_m\}$

$$e_j = \sum_{i=1}^m z_{ij} a_i, \quad \exists z_{ij}, 1 \leq i \leq m$$

$$\Rightarrow e_j = A z_j, \quad z_j = (z_{1j}, \dots, z_{mj})^T$$

$$\text{So, } [e_1 | e_2 | \dots | e_m] = A [z_1 | z_2 | \dots | z_m]$$

$$\Leftrightarrow \underbrace{I}_{m \times m \text{ identity matrix}} = A Z$$

Such matrix  $Z \in \mathbb{R}^{m \times m}$  is called the inverse of  $A$  and written  $A^{-1}$ .

Any nonsingular matrix has a unique inverse, and  $AA^{-1} = A^{-1}A = I$ .

### Thm (Equivalences of a nonsingular matrix)

For  $A \in \mathbb{R}^{m \times m}$ , the following statements are equivalent:

- (a)  $A$  has an inverse  $A^{-1}$
- (b)  $\text{rank}(A) = m$
- (c)  $\text{range}(A) = \mathbb{R}^m$
- (d)  $\text{null}(A) = \{0\}$
- (e)  $0$  is not an eigenvalue of  $A$
- (f)  $0$  is not a singular value of  $A$
- (g)  $\det(A) \neq 0$ .

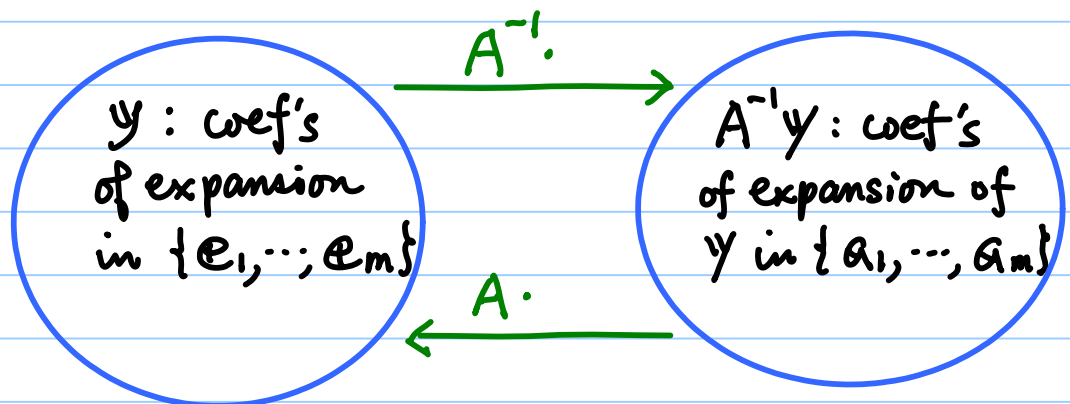
★ Matrix  $A^{-1}$  x vector

$$y = A x, \quad A: \text{nonsingular}$$

$$\Rightarrow x = A^{-1} y.$$

This means that  $A^{-1} y$  represents an expansion coefficients of  $y$  in the basis of col's of  $A$ .

So, Multiplication by  $A^{-1}$  is a change of basis operation!



Note:  $y = I^{-1} y$ .