Projection with Bases

Note Title 4/19/2012

★ Projection with an Orthonormal Basis

In general, for a projector $P \in \mathbb{R}^{m\times m}$

$$\dim(\text{range}(P)) = n \le m$$

So, let's define the matrix

$$\hat{Q} := [\,q_1 \cdots q_n\,] \in \mathbb{R}^{m\times n}$$

signifies the reduced form of Q

where $\{q_1, \cdots, q_n\}$ forms an O.N.B. of range(P) as in the proof of the previous theorem.

Then, $P = \hat{Q}\hat{Q}^T$ $\left(= Q\Lambda Q^T \text{ in the previous thm.} \right)$

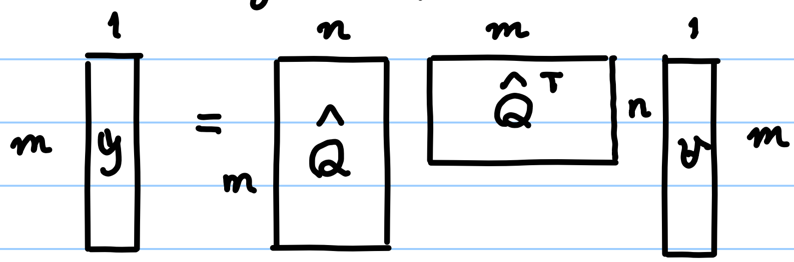
Recall $\forall v \in \mathbb{R}^m$, $\exists\, r \in \mathbb{R}^m$ (residual) s.t.

$$v = r + \sum_{i=1}^{n} (q_i q_i^T) v$$

the mapping

Hence, $v \longmapsto \sum_{i=1}^{n} (q_i q_i^T) v = y$

is an orthogonal projection onto range$(\hat{Q})$



as $n \to m$, $y \to v$ since $\hat{Q}\hat{Q}^T \to I$

Note: The complementary proj. to an orth. proj. $P = \hat{Q}\hat{Q}^T$ is also an orth. proj.

why? $\quad I - P = I - \hat{Q}\hat{Q}^T$ is also symmetric! //

Also note the following **special case**:

The rank-one orth. proj. with a unit vector $q \in \mathbb{R}^m$

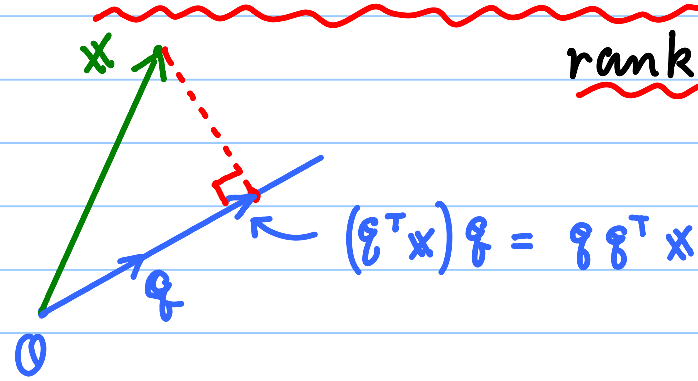
$$P_q := q q^T \in \mathbb{R}^{m \times m}$$

↳ a special rank 1 matrix

Its complementary proj. is

$$P_{\perp q} := I - P_q =: P_q^{\perp}$$

rank $m-1$ orth. proj.



For a general vector $a \in \mathbb{R}^m$ with $a \neq 0$, $\|a\| \neq 1$, the orth. proj. onto span$\{a\}$ becomes

$$P_a := \frac{a a^T}{a^T a} \rightarrow P_{\perp a} = I - \frac{a a^T}{a^T a}$$

why? Set $q = a/\|a\|$ then it's easy to show. //

☆ Projection with an Arbitrary Basis

Let $\{a_1, \cdots, a_n\} \subset \mathbb{R}^m$ be a set of linearly independent vectors.

Set $A = [a_1 \cdots a_n] \in \mathbb{R}^{m\times n}$

assume $m \geq n$.

Then what is the ortho. proj. onto $\text{range}(A) = \text{span}\{a_1, \cdots, a_n\}$?

Let $P_A \in \mathbb{R}^{m\times m}$ be such an ortho. proj.

Then $\text{range}(P_A) = \text{range}(A)$ and $\text{null}(P_A) = \text{null}(A)$.

Now take any $v \in \mathbb{R}^m$. Then $\exists\, v_1 \in \text{range}(A)$, $\exists\, v_2 \in \text{null}(A)$.

s.t. $v = v_1 + v_2$.

$\exists\, x \in \mathbb{R}^n$ s.t. $v_1 = Ax \perp v_2$

Hence $a_j \perp v_2 = v - v_1 = v - Ax$ for $1 \leq j \leq n$

$\iff a_j^T(v - Ax) = 0 \quad 1 \leq j \leq n$

$\iff A^T(v - Ax) = 0$

$\iff A^TAx = A^Tv$ (normal eqn. !!)

HW03 Prob. 6 { Since A is full rank (because $\{a_1, \cdots, a_n\}$ are lin. indep.), $(A^TA)^{-1}$ exists, i.e., $x = (A^TA)^{-1}A^Tv$

Hence $P_A = A(A^TA)^{-1}A^T$

We recover the previous projectors by setting

$$\begin{cases} A = \hat{Q} \Rightarrow \hat{Q}(\underbrace{\hat{Q}^T\hat{Q}}_{=I.})^{-1}\hat{Q}^T = \hat{Q}\hat{Q}^T \\ A = \underbrace{[a]}_{m\times 1 \text{ matrix}} \Rightarrow a(\underbrace{a^Ta}_{\text{a scalar}})^{-1}a^T = \dfrac{aa^T}{a^Ta} \end{cases}$$

Ex. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

(1) Compute the orthogonal projector P_A onto range (A)

(2) Let $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Then compute its projection onto range (A) using P_A obtained in (1).

<u>Sol.</u> (1) $P_A = A(A^TA)^{-1}A^T$

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

So, $P_A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$ //

(2) $P_A v = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ //

QR Decomposition / Factorization

★ Reduced QR Factorization

Let $A = [a_1 \cdots a_n] \in \mathbb{R}^{m\times n}$

Denote $\text{span}\{a_1, \cdots, a_k\}$ by $\langle a_1, \cdots, a_k \rangle$ for simplicity.

Then $\langle a_1 \rangle \subset \langle a_1, a_2 \rangle \subset \langle a_1, a_2, a_3 \rangle \subset \cdots \subset \langle a_1, \cdots, a_n \rangle$

successive column spaces.

QR factorization (or decomposition)
:= a method to successively construct a sequence of orthonormal vectors $q_1, q_2, \cdots$, s.t.
$\langle q_1, \cdots, q_j \rangle = \langle a_1, \cdots, a_j \rangle$
$j = 1, 2, \cdots, n$

If this is the case, then each a_j can be expanded by $\{q_1, \cdots, q_j\}$

Say, $\begin{cases} a_1 = r_{11} q_1 \\ a_2 = r_{12} q_1 + r_{22} q_2 \\ \vdots \\ a_n = r_{1n} q_1 + r_{2n} q_2 + \cdots + r_{nn} q_n \end{cases}$

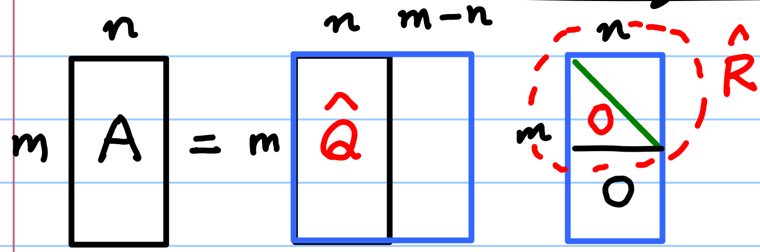
$$\Rightarrow A = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & \cdots & r_{2n} \\ & 0 & \ddots & \vdots \\ & & & r_{nn} \end{bmatrix}$$

$$= \hat{Q} \hat{R}$$

Reduced QR factorization (of A)

upper triangular

✩ Full QR Factorization



Append $m-n$ O.N. vectors to col's of $\hat{Q} \to Q$

Append $m-n$ 0's to rows of $\hat{R} \to R$

$$A = \underbrace{Q}_{m} \; R$$

<u>Note</u>: $q_j \perp \operatorname{range}(A)$ for $j > n$.

i.e., $\langle q_1, \cdots, q_n \rangle = \operatorname{range}(A)$

$$\langle q_{n+1}, \cdots, q_m \rangle = \operatorname{range}(A)^{\perp}$$

$$= \operatorname{null}(A^T)$$

✩ The Classical Gram-Schmidt Orthogonalization as QR

You must be familiar with the classical GS procedure.

Given $\{a_1, \cdots, a_n\}$, $a_j \in \mathbb{R}^m$, $1 \le j \le n$, construct an orthonormal set $\{q_1, \cdots, q_n\}$, $q_j \in \mathbb{R}^m$, $1 \le j \le n$ as follows:

$$q_1 = \frac{a_1}{r_{11}}, \quad r_{11} = \| a_1 \|$$

$$q_2 = \frac{a_2 - r_{12} q_1}{r_{22}}, \quad r_{12} = q_1^T a_2, \quad r_{22} = \| a_2 - r_{12} q_1 \|$$

$$q_3 = \frac{a_3 - r_{13}q_1 - r_{23}q_2}{r_{33}}, \qquad r_{i3} = q_i^T a_3,\ i=1,2.$$
$$r_{33} = \| a_3 - r_{13}q_1 - r_{23}q_2 \|$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$q_n = \frac{a_n - \sum_{i=1}^{n-1} r_{in} q_i}{r_{nn}}, \qquad r_{nn} = \| a_n - \sum_{i=1}^{n-1} r_{in} q_i \|$$

So, in general,

$$\begin{cases} r_{ij} = q_i^T a_j & i \neq j \\ r_{jj} = \| a_j - \sum_{i=1}^{j-1} r_{ij} q_i \| \end{cases}$$

<u>Algorithm (The classical Gram-Schmidt)</u> (<u>CGS</u>)

for $j = 1:n$
- $v_j = a_j$
- for $i = 1:j-1$
  - $r_{ij} = q_i^T a_j$
  - $v_j = v_j - r_{ij} q_i$ ← error accumulates here
- $r_{jj} = \| v_j \|$
- $q_j = v_j / r_{jj}$

Note the order of r_{ij} computation

$$\begin{array}{lccccc}
j: & ① & ② & ③ & & ⓝ \\
 & r_{11} & r_{12} & r_{13} & \cdots & r_{1n} \\
 & & r_{22} & r_{23} & & r_{2n} \\
 & & & r_{33} & & r_{3n} \\
 & & & & & \vdots \\
 & & & & & r_{nn}
\end{array}$$

Unfortunately, this version is numerically unstable.

Ex. $$A = \begin{bmatrix} 1 & 1 & 1 \\ \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}.$$ ε : small s.t. ε^2 can be ignored.

Apply CGS.

$$\begin{cases} r_{11} = \|a_1\| = \sqrt{1^2+\varepsilon^2+0^2+0^2} \approx 1. \\ q_1 = \dfrac{a_1}{r_{11}} = a_1. \end{cases}$$

$$\begin{cases} r_{12} = q_1^T a_2 = \begin{bmatrix} 1 & \varepsilon & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \varepsilon \\ 0 \end{bmatrix} = 1. \\ q_2 = \dfrac{a_2 - r_{12} q_1}{r_{22}} = \dfrac{1}{r_{22}} \left(\begin{bmatrix} 1 \\ 0 \\ \varepsilon \\ 0 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 \\ \varepsilon \\ 0 \\ 0 \end{bmatrix} \right) = \dfrac{1}{r_{22}} \begin{bmatrix} 0 \\ -\varepsilon \\ \varepsilon \\ 0 \end{bmatrix} \\ r_{22} = \|a_2 - r_{12} q_1\| = \varepsilon \sqrt{0+(-1)^2+1^2+0^2} = \sqrt{2}\,\varepsilon \\ q_2 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \end{cases}$$

$$r_{13} = q_1^T a_3 = 1, \quad r_{23} = q_2^T a_3 = 0.$$

$$q_3 = \frac{a_3 - r_{13} q_1 - r_{23} q_2}{r_{33}} = \frac{1}{r_{33}} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ \varepsilon \end{bmatrix} - \begin{bmatrix} 1 \\ \varepsilon \\ 0 \\ 0 \end{bmatrix} \right)$$

$$= \frac{1}{r_{33}} \begin{bmatrix} 0 \\ -\varepsilon \\ 0 \\ \varepsilon \end{bmatrix} = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad \text{with } r_{33} = \sqrt{2}\,\varepsilon.$$

Hence

$$\hat{Q} = \begin{bmatrix} 1 & 0 & 0 \\ \varepsilon & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}, \quad \hat{R} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \sqrt{2}\varepsilon & 0 \\ 0 & 0 & \sqrt{2}\varepsilon \end{bmatrix}$$

Let's check these results.

$$\hat{Q}\hat{R} = \begin{bmatrix} 1 & 1 & 1 \\ \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{bmatrix} = A$$

So, looks OK.

But, how about the orthogonality of $\hat{Q}$?

$$\hat{Q}^T\hat{Q} = \begin{bmatrix} 1 & \varepsilon & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \varepsilon & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1+\varepsilon^2 & -\varepsilon/\sqrt{2} & -\varepsilon/\sqrt{2} \\ -\varepsilon/\sqrt{2} & 1 & 1/2 \\ -\varepsilon/\sqrt{2} & 1/2 & 1 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & -\varepsilon/\sqrt{2} & -\varepsilon/\sqrt{2} \\ -\varepsilon/\sqrt{2} & 1 & 1/2 \\ -\varepsilon/\sqrt{2} & 1/2 & 1 \end{bmatrix}$$

$$\neq I_{3\times 3}$$

This is called "loss of orthogonality".