

Examples for Householder Triang. & Givens Rotations

Note Title

4/29/2012

Let's consider the following matrix:

$$A = \begin{bmatrix} 1 & 19 \\ -2 & -5 \\ 2 & 8 \end{bmatrix}$$

- Let's compute the QR factorization of A using the Householder Triang.

First of all, let's compute

$$\begin{aligned} v_1 &= \text{sign}(a_{11}) \|a_{1\cdot}\| e_1 + a_{1\cdot} \\ &= \text{sign}(1) \sqrt{1+(-2)^2+2^2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \\ &= +1 \times \sqrt{9} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \end{aligned}$$

$$Q_1 = F_1 = I_3 - 2 \frac{v_1 v_1^T}{v_1^T v_1} = I - 2 \frac{v_1 v_1^T}{v_1^T v_1}$$

$$v_1^T v_1 = 2^2 [2 \ -1 \ 1] \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 24$$

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{24} 2^2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} [2 \ -1 \ 1]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix}$$

$$\begin{aligned}
 Q_1 A &= \frac{1}{3} \begin{bmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 19 \\ -2 & -5 \\ 2 & 8 \end{bmatrix} \\
 &= \frac{1}{3} \begin{bmatrix} -9 & -45 \\ 0 & 36 \\ 0 & -27 \end{bmatrix} = \begin{bmatrix} -3 & -15 \\ 0 & 12 \\ 0 & -9 \end{bmatrix} \\
 &= (-3) \cdot \begin{bmatrix} 1 & 5 \\ 0 & -4 \\ 0 & 3 \end{bmatrix}
 \end{aligned}$$

Now our target is \downarrow this part and want to make it as $\begin{bmatrix} * \\ 0 \end{bmatrix}$.

$$\begin{aligned}
 v_2 &= \text{sign}(-4) \cdot \left\| \begin{bmatrix} -4 \\ 3 \end{bmatrix} \right\| \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ 3 \end{bmatrix} \\
 &= -1 \cdot 5 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -9 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix}
 \end{aligned}$$

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & F_2 \\ 0 & 0 \end{bmatrix}, \quad F_2 = I_2 - 2 \frac{v_2 v_2^T}{v_2^T v_2}$$

$$v_2^T v_2 = 3^2 \cdot [-3 \ 1] \begin{bmatrix} -3 \\ 1 \end{bmatrix} = 90$$

$$\begin{aligned}
 F_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{2}{90} \cdot 3^2 \begin{bmatrix} -3 \\ 1 \end{bmatrix} \begin{bmatrix} -3 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{5} \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix} \\
 &= \frac{1}{5} \begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix}
 \end{aligned}$$

$$\text{So, } Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{4}{5} & \frac{3}{5} \\ 0 & \frac{3}{5} & \frac{4}{5} \end{bmatrix}$$

$$\begin{aligned} \underbrace{Q_2 Q_1}_= Q^T A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{4}{5} & \frac{3}{5} \\ 0 & \frac{3}{5} & \frac{4}{5} \end{bmatrix} \cdot (-3) \begin{bmatrix} 1 & 5 \\ 0 & -4 \\ 0 & 3 \end{bmatrix} \\ &= (-3) \cdot \begin{bmatrix} 1 & 5 \\ 0 & 5 \\ 0 & 0 \end{bmatrix} \\ &= R \end{aligned}$$

$$\text{Now, } Q = (Q_2 Q_1)^T = Q_1^T Q_2^T$$

$$= Q_1 Q_2$$

$$= \frac{1}{3} \begin{bmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{4}{5} & \frac{3}{5} \\ 0 & \frac{3}{5} & \frac{4}{5} \end{bmatrix}$$

$$= \frac{1}{15} \begin{bmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & -4 & 3 \\ 0 & 3 & 4 \end{bmatrix}$$

$$= \frac{1}{15} \begin{bmatrix} -5 & -14 & -2 \\ 10 & -5 & 10 \\ -10 & 2 & 11 \end{bmatrix}$$

Note: As we noted before,
if you don't need Q , then
the computation becomes simpler.
For example, to apply Q_1 to A ,
it's easier to do the following:

$$\begin{aligned} Q_1 A &= Q_1 [a_1, a_2] = [Q_1 a_1, Q_1 a_2] \\ &= [(I - 2P_{v_1}) a_1, (I - 2P_{v_1}) a_2] \\ &= [a_1 - 2P_{v_1} a_1, a_2 - 2P_{v_1} a_2] \end{aligned}$$

Now,

$$\begin{aligned} P_{v_1} a_j &= \frac{v_1 v_1^T}{v_1^T v_1} a_j \\ \text{just an inner product} &\rightarrow \frac{v_1^T a_j}{v_1^T v_1} v_1, \quad j=1, 2 \end{aligned}$$

→ so, this is just a
constant multiple of v_1 .

→ So this becomes vector subtractions!

- Let's try the Givens Rotations!

$$A = \begin{bmatrix} \textcircled{1} & 19 \\ \textcircled{-2} & -5 \\ 2 & 8 \end{bmatrix}$$

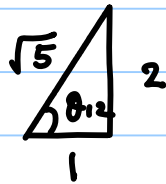
$\textcircled{1}$ serves as x_i , $\textcircled{-2}$ serves as x_j

$$G(1, 2, \theta_{12}) = \begin{bmatrix} \cos \theta_{12} & -\sin \theta_{12} & 0 \\ \sin \theta_{12} & \cos \theta_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\tan \theta_{12} = -x_j/x_i = 2.$$

$$\text{So, } \cos \theta_{12} = 1/\sqrt{5}$$

$$\sin \theta_{12} = 2/\sqrt{5}.$$

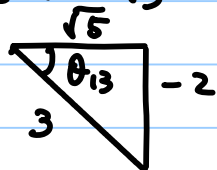


$$\text{Hence } G(1, 2, \theta_{12}) = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} G(1, 2, \theta_{12}) A &= \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 19 \\ -2 & -5 \\ 2 & 8 \end{bmatrix} \\ &= \begin{bmatrix} \textcircled{\sqrt{5}} & 29/\sqrt{5} \\ 0 & 33/\sqrt{5} \\ \textcircled{2} & 8 \end{bmatrix} \end{aligned}$$

$$\text{Now, } G(1, 3, \theta_{13}) = \begin{bmatrix} \cos \theta_{13} & 0 & -\sin \theta_{13} \\ 0 & 1 & 0 \\ \sin \theta_{13} & 0 & \cos \theta_{13} \end{bmatrix}$$

$$\tan \theta_{13} = -x_j/x_i = -2/\sqrt{5}$$



$$\cos \theta_{13} = \sqrt{5}/3$$

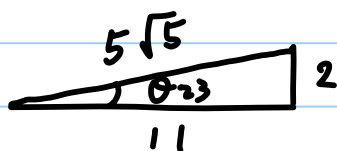
$$\sin \theta_{13} = -2/3$$

$$G(1,3,\theta_{13}) = \begin{bmatrix} \sqrt{5}/3 & 0 & +2/3 \\ 0 & 1 & 0 \\ -2/3 & 0 & \sqrt{5}/3 \end{bmatrix}$$

$$\begin{aligned} & G(1,3,\theta_{13}) G(1,2,\theta_{12}) A \\ &= \begin{bmatrix} \sqrt{5}/3 & 0 & +2/3 \\ 0 & 1 & 0 \\ -2/3 & 0 & \sqrt{5}/3 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 29/\sqrt{5} \\ 0 & 33/\sqrt{5} \\ 2 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 15 \\ 0 & 33/\sqrt{5} \\ 0 & -18\sqrt{5}/15 \end{bmatrix} \end{aligned}$$

$$G(2,3,\theta_{23}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & -\sin\theta_{23} \\ 0 & \sin\theta_{23} & \cos\theta_{23} \end{bmatrix}$$

$$\tan\theta_{23} = -x_j/x_i = \frac{18\sqrt{5}}{15} \cdot \frac{\sqrt{5}}{33} = \frac{2}{11}$$



$$\cos\theta_{23} = \frac{11}{5\sqrt{5}}$$

$$\sin\theta_{23} = \frac{2}{5\sqrt{5}}$$

$$G(2,3,\theta_{23}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 11/5\sqrt{5} & -2/5\sqrt{5} \\ 0 & 2/5\sqrt{5} & 11/5\sqrt{5} \end{bmatrix}$$

$$\begin{aligned} & G(2,3,\theta_{23}) G(1,3,\theta_{13}) G(1,2,\theta_{12}) A \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 11/5\sqrt{5} & -2/5\sqrt{5} \\ 0 & 2/5\sqrt{5} & 11/5\sqrt{5} \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 0 & 33/\sqrt{5} \\ 0 & -18\sqrt{5}/15 \end{bmatrix} = \begin{bmatrix} 3 & 15 \\ 0 & 15 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

$$Q = (G(2,3,\theta_{23}) G(1,3,\theta_{13}) G(1,2,\theta_{12}))^T \quad R //$$