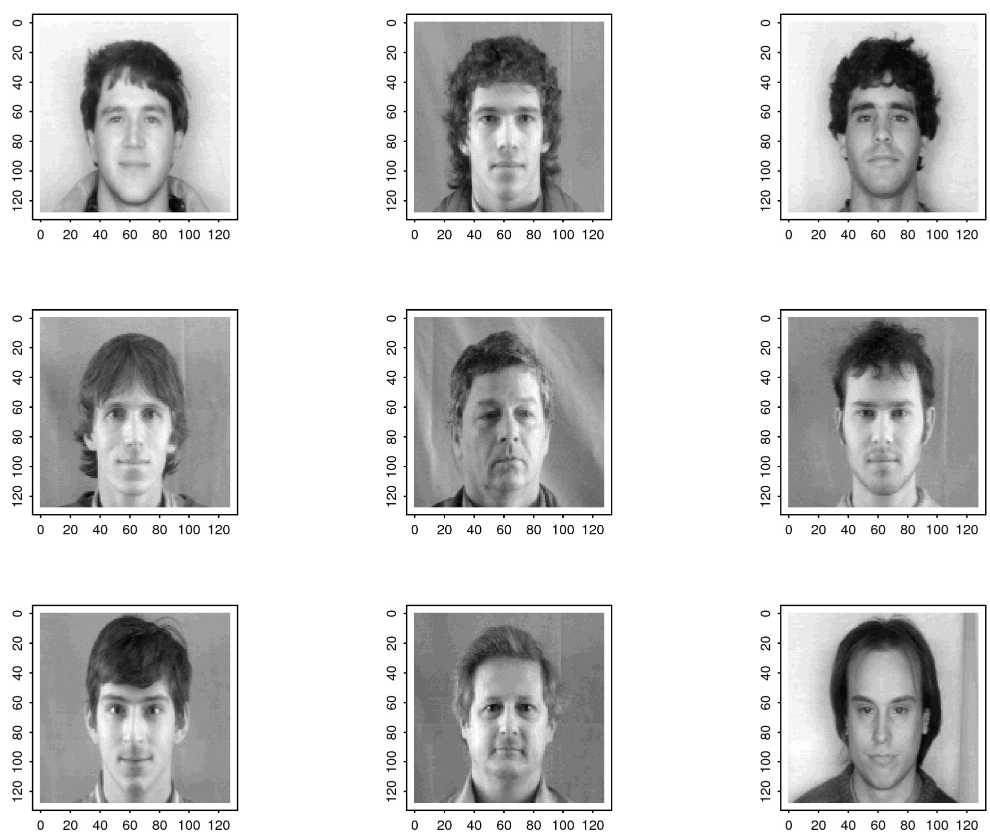
Continuation of Examples

Example 2: Face Image Database

- Provided by Prof. L. Sirovich (Mt Sinai S. M.)
- Consists of 143 faces of male caucasian students (and some faculty) at Brown Univ., without glasses, mustache, beard
- Each face is a gray-scale image with 128×128 pixels
- Horizontal dilation was applied so that the pupils are placed on two fixed points.

Below, 9 out of 143 are displayed:



Note that each image of size 128×128 pixels can be represented as a matrix of 128 rows and 128 columns or a vector of length $128 \times 128 = 16384$.

In fact, in MATLAB, if

$$X = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,128} \\ \vdots & \vdots & & \vdots \\ x_{128,1} & x_{128,2} & \cdots & x_{128,128} \end{bmatrix} \in \mathbb{R}^{128 \times 128}$$

then $X(:)$ represents a vector of length 16384 as follows:

$$X(:) \leftrightarrow \begin{bmatrix} x_{1,1} \\ \vdots \\ x_{128,1} \\ x_{1,2} \\ \vdots \\ x_{128,2} \\ \vdots \\ x_{1,128} \\ \vdots \\ x_{128,128} \end{bmatrix}$$

By default, a vector is a column vector

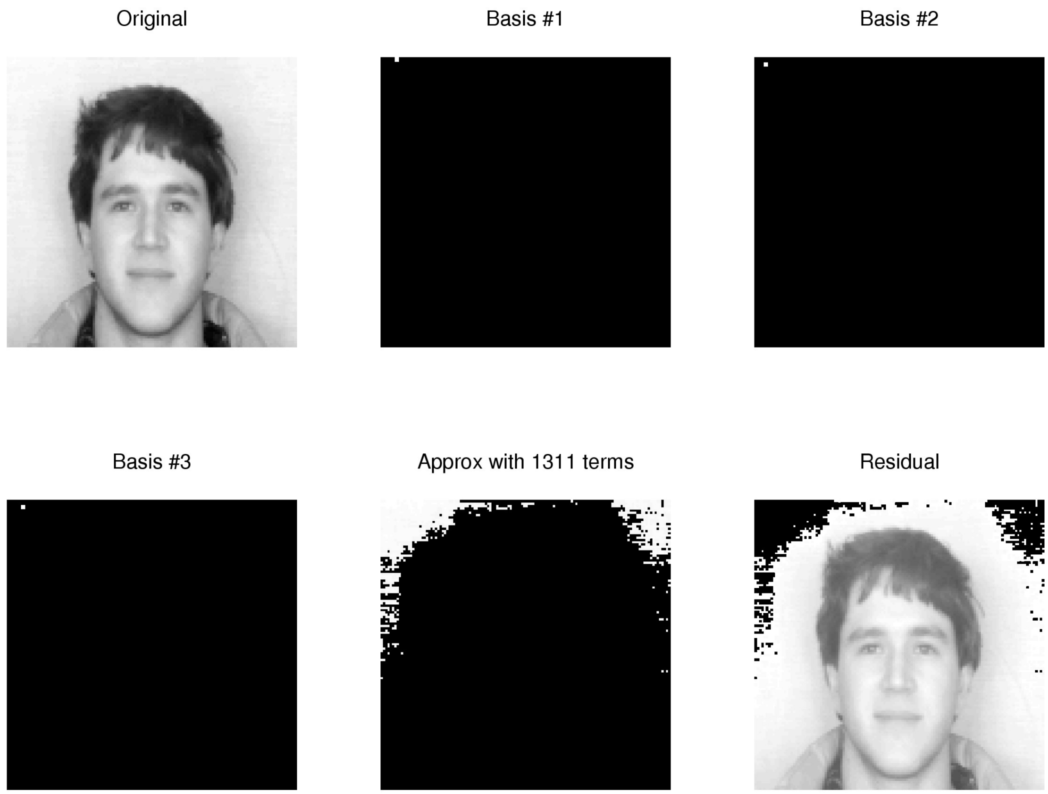
Now, let's pick one face image, and consider its representations using two different bases:

1) The standard basis $\{e_1, \cdots, e_n\}$
2) The wavelet basis $\{w_1, \cdots, w_n\}$

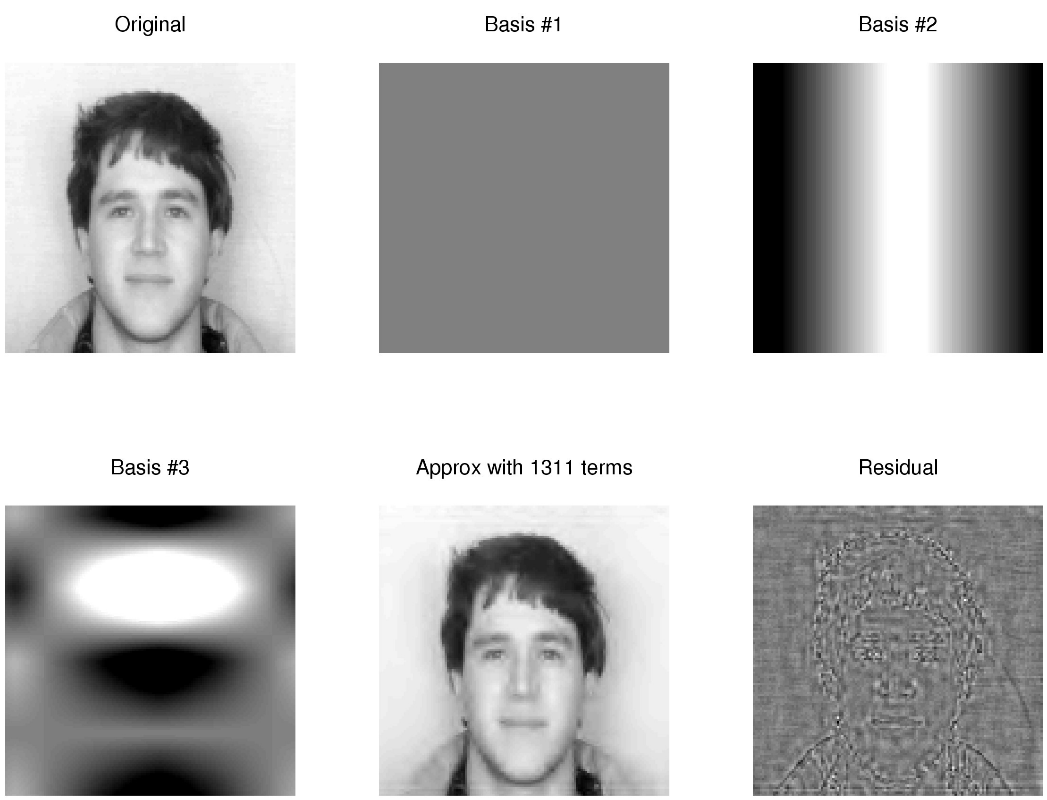
$n = 128^2 = 16384$

its relatives are used in JPEG2000!

1) The standard basis



2) The wavelet basis



Example 3: Term-Document Matrices for Search

This example is from Berry & Browne: "Understanding Search Engines", 2005.

Consider a set of documents consisting of book titles, and a set of words (or terms) as follows:

Terms		*Documents*	
T1:	Bab(y,ies,y's)	D1:	Infant & Toddler First Aid
T2:	Child(ren's)	D2:	Babies & Children's Room (For Your Home)
T3:	Guide	D3:	Child Safety at Home
T4:	Health	D4:	Your Baby's Health and Safety: From Infant to Toddler
T5:	Home	D5:	Baby Proofing Basics
T6:	Infant	D6:	Your Guide to Easy Rust Proofing
T7:	Proofing	D7:	Beanie Babies Collector's Guide
T8:	Safety		
T9:	Toddler		

Then, consider the following term-document matrix of size 9×7.

The 9×7 term-by-document matrix before normalization, where the element $\hat{a}_{ij}$ is the number of times term i appears in document title j:

$$\text{Terms}\left\{\begin{array}{r} \\ T2 \\ \\ \\ \\ \\ T7 \\ \\ \\ \end{array}\right. \quad \hat{A} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}}_{\text{documents}}$$

D2 D3 D5 D6

← rather sparse!

Now suppose we want to retrieve books on "Child Proofing"

T2 T7

Then this can be reprented by a query vector:

$$q = \begin{array}{c} \left[\begin{array}{ccccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]^T \\ \begin{array}{ccccccccc} & T2 & & & & & T7 & & \end{array} \end{array}$$

Try to match q with each column vector (i.e., a document).

$\Rightarrow$ No exact hit, but the close ones are D2, D3, D5, D6

Generalizing this to:

$\left\{ \begin{array}{l} \text{Terms} = \text{the English dictionary} \\ \text{documents} = \text{the entire webpages} \end{array} \right.$

the term-document matrix becomes huge!!

est. in 2017

300,000 × 47,000,000,000

or even larger now!

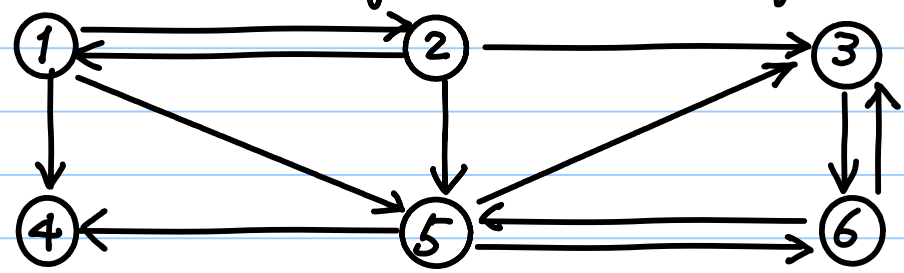
How to deal with such a huge matrix for search?

$\Rightarrow$ We'll learn how later in this course.

Example 4: Link Graph Matrix of webpages

Consider a very small set of webpages

6 web pages



This graph means that

① has outlinks to ②, ④, ⑤
(pointers)

① has an inlink from ②

② has ---

A link graph matrix $P = (P_{ij})$ is defined by

$$P_{ij} = \begin{cases} \text{nonzero} & \text{if } \exists \text{ ⓙ} \to \text{ⓘ} \\ 0 & \text{otherwise} \end{cases}$$

there exists

The nonzero const. is determined by the normalization

$$\sum_{i=1}^{n} P_{ij} = 1 \quad \text{for } 1 \le j \le n$$

So, in this small example, we have

$$P = \begin{bmatrix} 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 & 1/3 & 1/2 \\ 1/3 & 0 & 0 & 0 & 1/3 & 0 \\ 1/3 & 1/3 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 1/3 & 0 \end{bmatrix}$$

//

One possible measure of importance of a webpage (i)
= how many other important pages have outlinks to (i)

- This can be solved by the eigenvalue decomposition of P
- In reality, $n >$ billions!
- Depending on the literature, P^T is denoted as "P", i.e.,
$$\sum_{j=1}^{n} P_{ij} = 1, \quad 1 \leq i \leq n$$

$\Rightarrow$ We'll discuss link graph matrices in detail later in this course.

Floating Point Computations

☆ Floating Point Numbers

On a digital computer, one can only use a finite number of bits to represent a real number, e.g., $\sqrt{2}$, π, e, etc.

Hence, the floating point representation was invented and standardized.
$\Rightarrow$ IEEE 754 Standard.

The idealized floating point system is a discrete subset $\mathbb{F} \subset \mathbb{R}$

$$\mathbb{F} := \{0\} \cup \{ x \in \mathbb{R} \mid x = \pm 1.d_1 d_2 \cdots d_t \times \beta^e, \\ d_j \in \{0, 1, \cdots, \beta - 1\},\ 1 \le j \le t \\ t \ge 1,\ t \in \mathbb{N}, \\ \beta \ge 2,\ \beta \in \mathbb{N}, \\ e \in \mathbb{Z} \}$$

$d_1 d_2 \cdots d_t$ = mantissa (or fraction)

β = base (or radix), typically $\beta = 2$.

e = exponent, t = precision.

$$1.d_1 d_2 \cdots d_t \times \beta^e \text{ in base } \beta$$

$$= \left(\frac{1}{\beta^0} + \frac{d_1}{\beta^1} + \frac{d_2}{\beta^2} + \cdots + \frac{d_t}{\beta^t} \right) \times \beta^e \text{ in base 10}$$

In C++, they are float, double, long double.

Basic f.p. types for a real number:

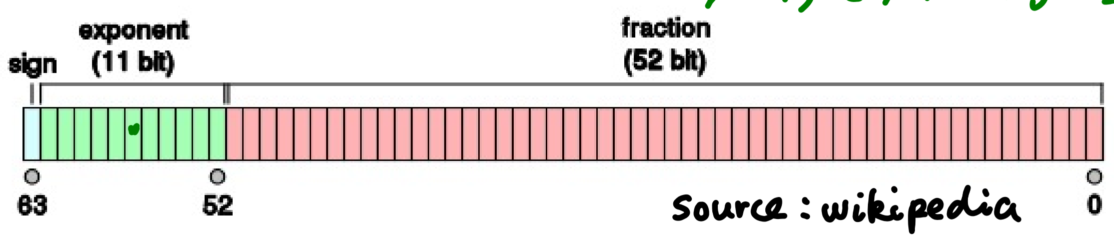
Single precision f.p. number = 32 bits

Double 〃 〃 = 64 bits

Quadraple 〃 〃 = 128 bits

MATLAB Default!

⇒ 4, 8, 16 bytes



source: wikipedia

How about t ?

In IEEE 754 Standard,

$$t = \begin{cases} 24 & \text{for single precision} \\ 53 & \text{double 〃} \\ 113 & \text{quadraple 〃} \end{cases}$$

$\mu := \frac{1}{2}\beta^{1-t}$ is called the machine epsilon or the unit round-off of the floating point system, often denoted by ε_{mach}.

Single Precision $\mu = 2^{-24} \approx 5.96 \times 10^{-8}$

Double Precision $\mu = 2^{-53} \approx 1.11 \times 10^{-16}$

Quadraple Precision $\mu = 2^{-113} \approx 9.63 \times 10^{-35}$

How about e ?

In IEEE 754 Standard,

$$\begin{cases} \text{single precision}: & -126 \leq e \leq 127 \\ \text{double} \quad " & : -1022 \leq e \leq 1023 \\ \text{quadraple} \quad " & : -16382 \leq e \leq 16383 \end{cases}$$

Hence the range of representable positive numbers are roughly:

$$\begin{cases} \text{single prec.}: & 10^{-38} \sim 10^{+38} \\ \text{double} \quad " : & 10^{-308} \sim 10^{+308} \\ \text{quadraple} \quad " : & 10^{-4932} \sim 10^{+4932} \end{cases}$$

Now, we have:

$$(*) \quad \forall x \in \mathbb{R}, \ \exists x' \in \mathbb{F} \ \text{ s.t. } \ |x - x'| \leq \mu |x|$$

Let's define the following function:

$fl : \mathbb{R} \to \mathbb{F}$

a fcn giving the closest f.p. approximation to a given real num.
= rounded equivalent in $\mathbb{F}$.

So, the statement (*) is equivalent to
$\forall x \in \mathbb{R}$, $\exists \varepsilon \in \mathbb{R}$ with $|\varepsilon| \leq \mu$ s.t.

$$fl(x) = x(1+\varepsilon)$$

Why is this so?

$$|x - fl(x)| \leq \mu |x|$$

$$\Leftrightarrow \quad -\mu|x| \leq x - fl(x) \leq \mu|x|$$

$$\Leftrightarrow \quad x - \mu|x| \leq fl(x) \leq x + \mu|x|$$

$$\Leftrightarrow \quad fl(x) = x + \varepsilon x \text{ for } \exists\, \varepsilon \text{ with } |\varepsilon| \leq \mu$$

//

In essence, on any computer, for any real number x, our best hope is $\underbrace{\left|\frac{x - fl(x)}{x}\right|}_{= \text{relative error}} \leq \mu = \varepsilon_{mach}$ $\approx 10^{-16}$ in MATLAB eps

Floating Point Arithmetic

Basic arithmetic operations

in $\mathbb{R} \Rightarrow +, -, \times, \div$

in $\mathbb{F} \Rightarrow \oplus, \ominus, \otimes, \oslash$

Ideal goal of computer arithmetic:

$$\forall x, y \in \mathbb{F}, \quad \underbrace{x \circledcirc y}_{\text{operations in } \mathbb{F}} = fl(\underbrace{x \circ y}_{\text{operations in } \mathbb{R}})$$

$\circ = +, -, \times, \text{ or } \div$

From the above discussion, we have

$$(\star) \quad x \circledcirc y = (x \circ y)(1 + \varepsilon), \ \exists\, |\varepsilon| < \mu$$

If a computer truncates (instead of rounds) the intermediate result $x \circ y$, then in (☆) μ should be replaced by 2μ.

Nick Trefethen (Oxford) calls (☆)
"The Fundamental Axiom of Floating Point Arithmetic."

☆ Floating Point Operations (FLOPS)

FLOPS = flops = flop/s
= f.p. operations/second
= a measure of a computer's performance

flop = a unit of one arithmetic operation on a computer

Ex. The statement in a code

$$y = y + a * x ;$$

How many flops does this require?

$\Rightarrow$ 2 flops

as of June 2016, it is Sunway TaihuLight (China)

Note: The world fastest computer can perform $\geq$ 93 Peta FLOPS $= 93 \times 10^{15}$ FLOPS!