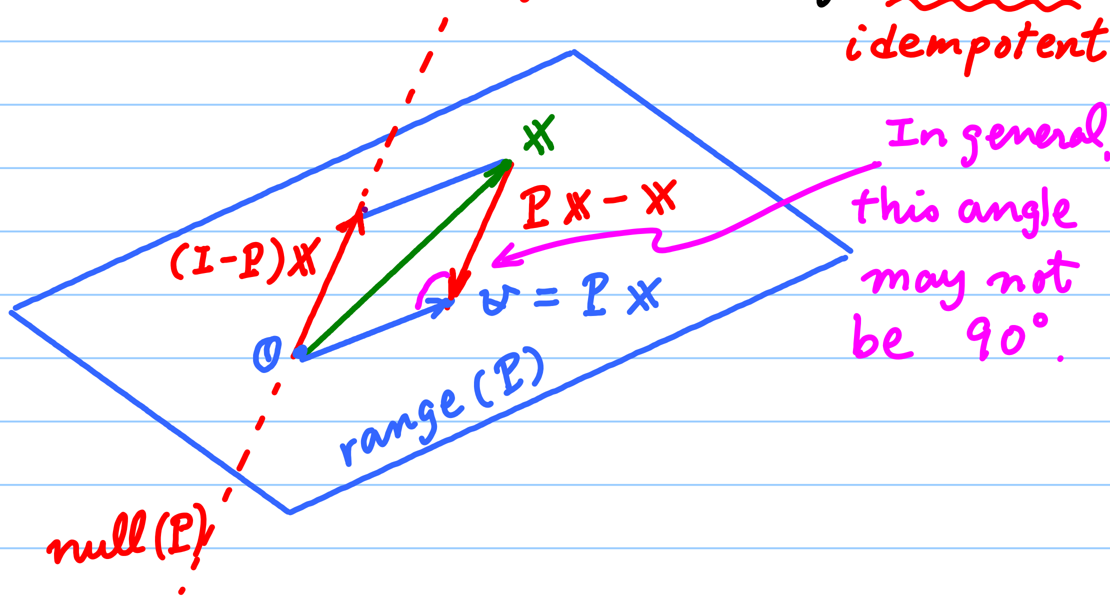
Projectors

Note Title

★ Projectors

Def. A matrix $P \in \mathbb{R}^{m\times m}$ is called a projector if $\underbrace{P^2 = P}_{\text{idempotent}}$



Let $v \in \text{range}(P)$

Then $\exists\, x \in \mathbb{R}^m$ s.t. $Px = v$

$\Rightarrow Pv = P(Px) = P^2 x = Px = v$

In other words, once $v \in \text{range}(P)$ then applying P to v does not change v.

("shadows remain as shadows.")

also, $\forall x \in \mathbb{R}^m$, $Px - x \in \text{null}(P)$

why? $P(Px - x) = P^2x - Px$
$= Px - Px = 0$ //

Def. Let $P \in \mathbb{R}^{m\times m}$ be a projector.

Then $I - P$ is also a projector and is called the complementary projector to P.

Let's check $I-P$ is a projector.

$$(I-P)^2 = (I-P)(I-P)$$
$$= I - P - P + P^2$$
$$= I - P \quad ✓$$

$I-P$ is a projector onto $\text{null}(P)$!

<u>Thm</u> $\text{range}(I-P) = \text{null}(P)$
$\text{null}(I-P) = \text{range}(P)$

i.e., P & $I-P$: really complementary!

(Proof) Take any $v \in \text{null}(P)$,
i.e., $Pv = 0$.

Then $\underline{(I-P)v} = v - Pv = \underline{v}$

i.e., $v \in \text{range}(I-P)$ because v is written as a matrix-vector product $(I-P)v$.

So, $\text{null}(P) \subset \text{range}(I-P)$ ✓

On the other hand,
take any $v \in \text{range}(I-P)$.
Then, $\exists\, x \in \mathbb{R}^m$ s.t.
$$v = (I-P)x$$

Apply P to both sides:
$$Pv = P(I-P)x$$
$$= (P - P^2)x = 0$$
i.e., $v \in \text{null}(P)$

So, $\text{range}(I-P) \subset \text{null}(P)$ ✓

Hence we have $\text{range}(I-P) = \text{null}(P)$ //

It's now easy to prove the other statement : $\text{null}(I-P) = \text{range}(P)$ by writing $\tilde{P} = I - P$ and repeat the above argument for $\tilde{P}$. ///

<u>Thm</u> $\text{null}(I-P) \cap \text{null}(P) = \{0\}$.

i.e., $\text{range}(P) \cap \text{null}(P) = \{0\}$.

(Proof) Take any $v \in \text{null}(I-P) \cap \text{null}(P)$

Then, $(I-P)v = 0$ & $Pv = 0$

$\Leftrightarrow v = 0$ ///

These theorems imply that

" A projector separates $\mathbb{R}^m$ into two spaces, i.e.,

$$\mathbb{R}^m = \text{range}(P) + \text{null}(P) \text{"}$$

In other words,

$\forall v \in \mathbb{R}^m$, $\exists\, v_1 \in \text{range}(P)$,

$\exists\, v_2 \in \text{null}(P)$, s.t.

$v = v_1 + v_2$

and this decomposition is unique for a given projector P.

<u>why?</u> Suppose this decomposition is not unique. Then $\exists\, \mathbf{x} \in \mathbb{R}^m$, $\mathbf{x} \neq \mathbf{0}$ s.t.

$$\mathbf{v} = \underbrace{(\mathbf{v}_1 + \mathbf{x})}_{\in\, \mathrm{range}(P)} + \underbrace{(\mathbf{v}_2 - \mathbf{x})}_{\in\, \mathrm{null}(P)}$$

But this means that

$$\mathbf{x} \in \mathrm{range}(P) \;\; \& \;\; \mathbf{x} \in \mathrm{null}(P)$$

i.e., $\mathbf{x} \in \underbrace{\mathrm{range}(P) \cap \mathrm{null}(P)}_{= \{\mathbf{0}\}}$.

$\Rightarrow \mathbf{x} = \mathbf{0}$. #

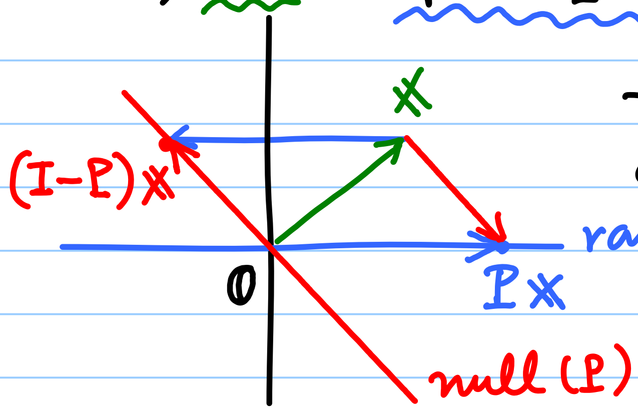
<u>a simple example</u>

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2} \qquad P^2 = P$$

$$I - P = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \qquad (I-P)^2 = I - P$$

$$\mathrm{range}(P) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$
$$\mathrm{null}(P) = \mathrm{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right),$$

So, $\mathbb{R}^2 = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + \mathrm{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$



This is not an orthogonal decomp.

★ Orthogonal Projectors

Def. A projector $P \in \mathbb{R}^{m\times m}$ is said to be **orthogonal** if range$(P) \perp$ null(P)

Ex. Consider $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ in $\mathbb{R}^2$

This is the orthogonal projector onto "x-axis". The complementary proj. is also orthogonal, i.e., orth. proj. onto "y-axis", and

$$\mathbb{R}^2 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \underset{\text{orthogonal}}{\oplus} \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$$

Note: Do not confuse an orthogonal projector P with an orthogonal matrix!

What happens if P is a projector and is an orthogonal matrix?

$P^2 = P$ (proj.); $P^T = P^{-1}$ (orth. mat.)

$$\underbrace{P^T P^2}_{=P} = \underbrace{P^T P}_{=I}. \qquad P^T P = I$$

$$\Rightarrow P = I \text{ //}$$

Thm A projector P is an orthogonal projector iff $P^T = P$, i.e., symmetric.

(Proof) $(\Leftarrow)$: Take any $v_1 \in \text{range}(P)$,
any $v_2 \in \text{null}(P)$.

Then $\exists\, x \in \mathbb{R}^m$ s.t., $v_1 = Px$.

$$\Rightarrow \quad v_1^T v_2 = (Px)^T v_2 = x^T P^T v_2$$

$$\xrightarrow{P^T = P} \; = x^T P v_2 = x^T 0 = 0.$$

i.e., $\text{range}(P) \perp \text{null}(P)$ ✓

$(\Rightarrow)$: (a bit more tough to show :-))

Since $\text{range}(P) \oplus \text{null}(P)$,
$\exists$ orthonormal basis (O.N.B.) of $\mathbb{R}^m$
$\{q_1, \cdots, q_m\}$ s.t.

$\text{range}(P) = \text{span}\{q_1, \cdots, q_n\}$

$\text{null}(P) = \text{span}\{q_{n+1}, \cdots, q_m\}$

Then,
$$P q_j = \begin{cases} q_j & \text{for } 1 \le j \le n \\ 0 & \text{for } n+1 \le j \le m \end{cases}$$

Let $Q = [q_1 \cdots\cdots q_m] \in \mathbb{R}^{m \times m}$

Then
$$PQ = Q \underbrace{\left[\begin{array}{c|c} I_{n \times n} & O_{n \times (m-n)} \\ \hline O_{(m-n) \times n} & O_{(m-n) \times (m-n)} \end{array}\right]}_{\text{call this } \Lambda}$$

i.e., $PQ = Q\Lambda$

Multiply Q^T from right.

This is a diagonal matrix

$$\Rightarrow \quad P \underbrace{QQ^T}_{= I} = Q \Lambda Q^T$$

So, $P = Q \Lambda Q^T$

$$\Rightarrow \quad P^T = (Q \Lambda Q^T)^T = (Q^T)^T \Lambda^T Q^T$$
$$= Q \Lambda Q^T = P \quad ///$$