Examples for Householder Triang. & Givens Rotations

Note Title

Let's consider the following matrix :

$$A = \begin{bmatrix} 1 & 19 \\ -2 & -5 \\ 2 & 8 \end{bmatrix}$$

• Let's compute the QR factorization of A using the Householder Triang.

First of all, let's compute

$$v_1 = \text{sign}(a_{11}) \|a_1\| e_1 + a_1$$

$$= \text{sign}(1) \sqrt{1+(-2)^2+2^2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$= +1 \times \sqrt{9} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$Q_1 = F_1 = I_3 - 2P_{v_1} = I - 2\frac{v_1 v_1^T}{v_1^T v_1}$$

$$v_1^T v_1 = 2^2 \begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 24$$

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{24} 2^2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix}$$

$$Q_1 A = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 19 \\ -2 & -5 \\ 2 & 8 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -9 & -45 \\ 0 & 36 \\ 0 & -27 \end{bmatrix} = \begin{bmatrix} -3 & -15 \\ 0 & 12 \\ 0 & -9 \end{bmatrix}$$

$$= (-3)\cdot \begin{bmatrix} 1 & 5 \\ 0 & \boxed{-4} \\ 0 & \boxed{3} \end{bmatrix}$$

Now our target is this part and want to make it as $\begin{bmatrix} * \\ 0 \end{bmatrix}$.

$$v_2 = \text{sign}(-4)\cdot \left\| \begin{bmatrix} -4 \\ 3 \end{bmatrix} \right\| \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$= -1\cdot 5\cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -9 \\ 3 \end{bmatrix} = 3\cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{F_2}} \\ 0 & & \end{bmatrix}, \quad F_2 = I_2 - 2\frac{v_2 v_2^T}{v_2^T v_2}$$

$$v_2^T v_2 = 3^2 \cdot [-3 \ \ 1]\begin{bmatrix} -3 \\ 1 \end{bmatrix} = 90$$

$$F_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{2}{90}\cdot 3^2 \begin{bmatrix} -3 \\ 1 \end{bmatrix}[-3 \ \ 1]$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix}$$

So,

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4/5 & 3/5 \\ 0 & 3/5 & 4/5 \end{bmatrix}$$

$$\underbrace{Q_2 Q_1}_{= Q^T} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4/5 & 3/5 \\ 0 & 3/5 & 4/5 \end{bmatrix} \cdot (-3) \begin{bmatrix} 1 & 5 \\ 0 & -4 \\ 0 & 3 \end{bmatrix}$$

$$= \underbrace{(-3) \cdot \begin{bmatrix} 1 & 5 \\ 0 & 5 \\ 0 & 0 \end{bmatrix}}_{= R}$$

Now, $Q = (Q_2 Q_1)^T = Q_1^T Q_2^T$

$$= Q_1 Q_2$$

$$= \frac{1}{3} \begin{bmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4/5 & 3/5 \\ 0 & 3/5 & 4/5 \end{bmatrix}$$

$$= \frac{1}{15} \begin{bmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & -4 & 3 \\ 0 & 3 & 4 \end{bmatrix}$$

$$= \frac{1}{15} \begin{bmatrix} -5 & -14 & -2 \\ 10 & -5 & 10 \\ -10 & 2 & 11 \end{bmatrix}$$

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Note: As we noted before, if you don't need Q, then the computation becomes simpler. For example, to apply Q_1 to A, it's easier to do the following:

$$Q_1 A = Q_1 [a_1 \;\; a_2] = [Q_1 a_1 \;\; Q_1 a_2]$$

$$= \left[(I - 2P_{v_1}) a_1 \;\; (I - 2P_{v_1}) a_2 \right]$$

$$= [a_1 - 2P_{v_1} a_1 \quad a_2 - 2P_{v_1} a_2]$$

Now,

$$P_{v_1} a_j = \frac{v_1 v_1^T}{v_1^T v_1} a_j$$

just an inner product

$$= \frac{v_1^T a_j}{v_1^T v_1} v_1 \quad , \quad j = 1, 2$$

→ so, this is just a constant multiple of v_1.

So this becomes vector subtractions!

- Let's try the Givens Rotations!

$$A = \begin{bmatrix} 1 & 19 \\ -2 & -5 \\ 2 & 8 \end{bmatrix}$$

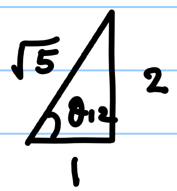
$\bigcirc$ serves as x_i, $\bigcirc$ serves as x_j

$$G(1,2,\theta_{12}) = \begin{bmatrix} \cos\theta_{12} & -\sin\theta_{12} & 0 \\ \sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\tan\theta_{12} = -x_j/x_i = 2.$$

So, $\cos\theta_{12} = 1/\sqrt{5}$

$\sin\theta_{12} = 2/\sqrt{5}$.



Hence $G(1,2,\theta_{12}) = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

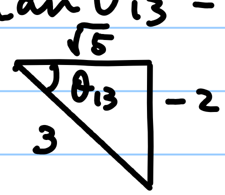
$$G(1,2,\theta_{12})\,A = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 19 \\ -2 & -5 \\ 2 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{5} & 29/\sqrt{5} \\ 0 & 33/\sqrt{5} \\ 2 & 8 \end{bmatrix}$$

Now,

$$G(1,3,\theta_{13}) = \begin{bmatrix} \cos\theta_{13} & 0 & -\sin\theta_{13} \\ 0 & 1 & 0 \\ \sin\theta_{13} & 0 & \cos\theta_{13} \end{bmatrix}$$

$$\tan\theta_{13} = -x_j/x_i = -2/\sqrt{5}$$



$\cos\theta_{13} = \sqrt{5}/3$

$\sin\theta_{13} = -2/3$

$$G(1,3,\theta_{13}) = \begin{bmatrix} \sqrt{5}/3 & 0 & +2/3 \\ 0 & 1 & 0 \\ -2/3 & 0 & \sqrt{5}/3 \end{bmatrix}$$

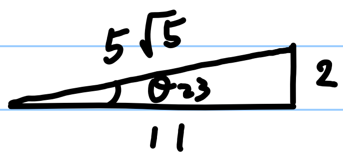
$$G(1,3,\theta_{13})\,G(1,2,\theta_{12})\,A$$

$$= \begin{bmatrix} \sqrt{5}/3 & 0 & +2/3 \\ 0 & 1 & 0 \\ -2/3 & 0 & \sqrt{5}/3 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 29/\sqrt{5} \\ 0 & 33/\sqrt{5} \\ 2 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 15 \\ 0 & 33/\sqrt{5} \\ 0 & -18\sqrt{5}/15 \end{bmatrix}$$

$$G(2,3,\theta_{23}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & -\sin\theta_{23} \\ 0 & \sin\theta_{23} & \cos\theta_{23} \end{bmatrix}$$

$$\tan\theta_{23} = -x_j/x_i = \frac{18\sqrt{5}}{15} \cdot \frac{\sqrt{5}}{33} = \frac{2}{11}$$



$$\cos\theta_{23} = \frac{11}{5\sqrt{5}}$$

$$\sin\theta_{23} = \frac{2}{5\sqrt{5}}$$

$$G(2,3,\theta_{23}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 11/5\sqrt{5} & -2/5\sqrt{5} \\ 0 & 2/5\sqrt{5} & 11/5\sqrt{5} \end{bmatrix}$$

$$G(2,3,\theta_{23})\,G(1,3,\theta_{13})\,G(1,2,\theta_{12})\,A$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 11/5\sqrt{5} & -2/5\sqrt{5} \\ 0 & 2/5\sqrt{5} & 11/5\sqrt{5} \end{bmatrix} \begin{bmatrix} 3 & 15 \\ 0 & 33/\sqrt{5} \\ 0 & -18\sqrt{5}/15 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 15 \\ 0 & 15 \\ 0 & 0 \end{bmatrix}}_{R}$$

$$Q = \left(G(2,3,\theta_{23})\,G(1,3,\theta_{13})\,G(1,2,\theta_{12})\right)^T$$