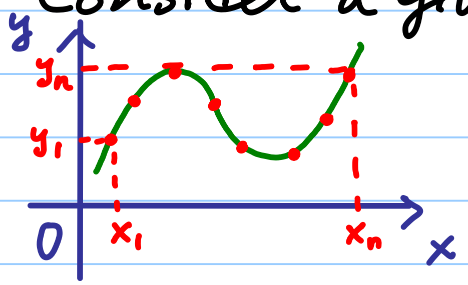
# Lecture 6: Applications of Calc. Variations to Interpolation & Approximation

Note Title

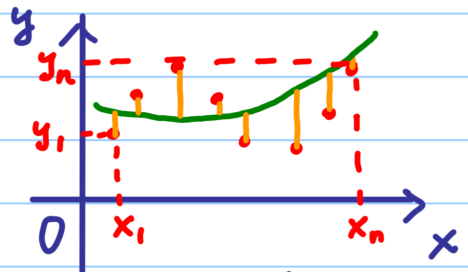
## ☆ Interpolation vs Approximation

Consider a given set of pts $\{(x_j, y_j)\}_{j=1:n} \subset \mathbb{R}^2$.




• **Interpolation:**
Find a fcn $y = f(x)$ s.t.
$y_j = f(x_j)$, $j = 1:n$

⇒ Once you find $f(x)$, you can evaluate at any pt $x$ ($\neq x_j$)




• **Approximation (smoothing):**
Find a fcn $y = f(x)$ s.t. maximizing the fidelity to the data (i.e., minimizing the residual error) subj. to some smoothness constraint on $f$. ⇒ This includes the least squares fit of a polynomial to the data, e.g.,

$$\sum_{j=1}^{n} (y_j - f(x_j))^2 \to \min!$$

subj. to $f \in \mathcal{P}_k$ ($k$ th order polynom.)
$k \leq n-1$

We'll mainly focus on interpolation here.

Note: $\exists$ vast literature on both interpol. & approx. Multidimensional cases are particularly important in geophysics, medicine, cartography, spatial statistics, and image processing, ---

As you can imagine, mathematicians started analyzing these problems using algebraic poly's, trigonometric poly's ⇒ Chebyshev Poly's, wavelets, rational fcns ---

Runge

Gibbs

## ☆ Lagrange Polynomials / The Runge Phenomenon

**Def.** (Lagrange polynomial)

Given a set of pts $\{(x_j, y_j)\}_{j=0:k}$ ($k+1$ pts)

let $L_k(x) := \sum_{j=0}^{k} y_j \, l_j(x)$ → $x_j \neq x_i$ if $j \neq i$

where $$l_j(x) := \prod_{\substack{0 \le m \le k \\ m \neq j}} \frac{x - x_m}{x_j - x_m}$$

$$= \frac{(x-x_0)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_k)}{(x_j-x_0)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_k)}$$

Hence, $l_j(x_m) = \delta_{jm} = \begin{cases} 1 & \text{if } m=j \\ 0 & \text{if } m \neq j \end{cases}$

↳ the Kronecker delta

→ This is a fundamental property for interpolation ( $L_k(x_j) = y_j$ )

**Note**: • $\deg(l_j) = k$.

• By defining $\Phi(x) := \prod_{j=0}^{k} (x - x_j)$,

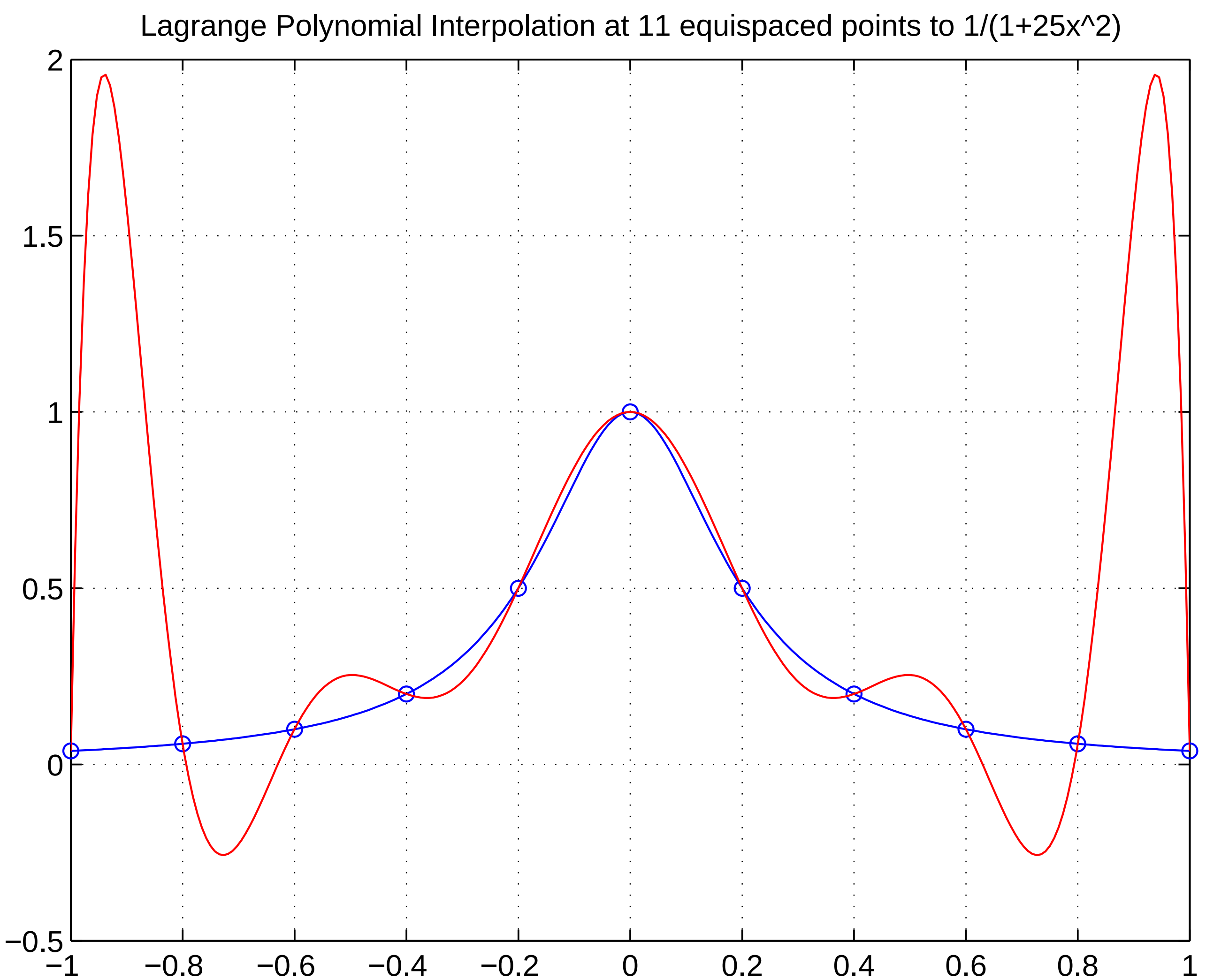
we can write $l_j(x) = \dfrac{\Phi(x)}{(x-x_j)\,\Phi'(x_j)}$

• The Runge Phenomenon (1901)

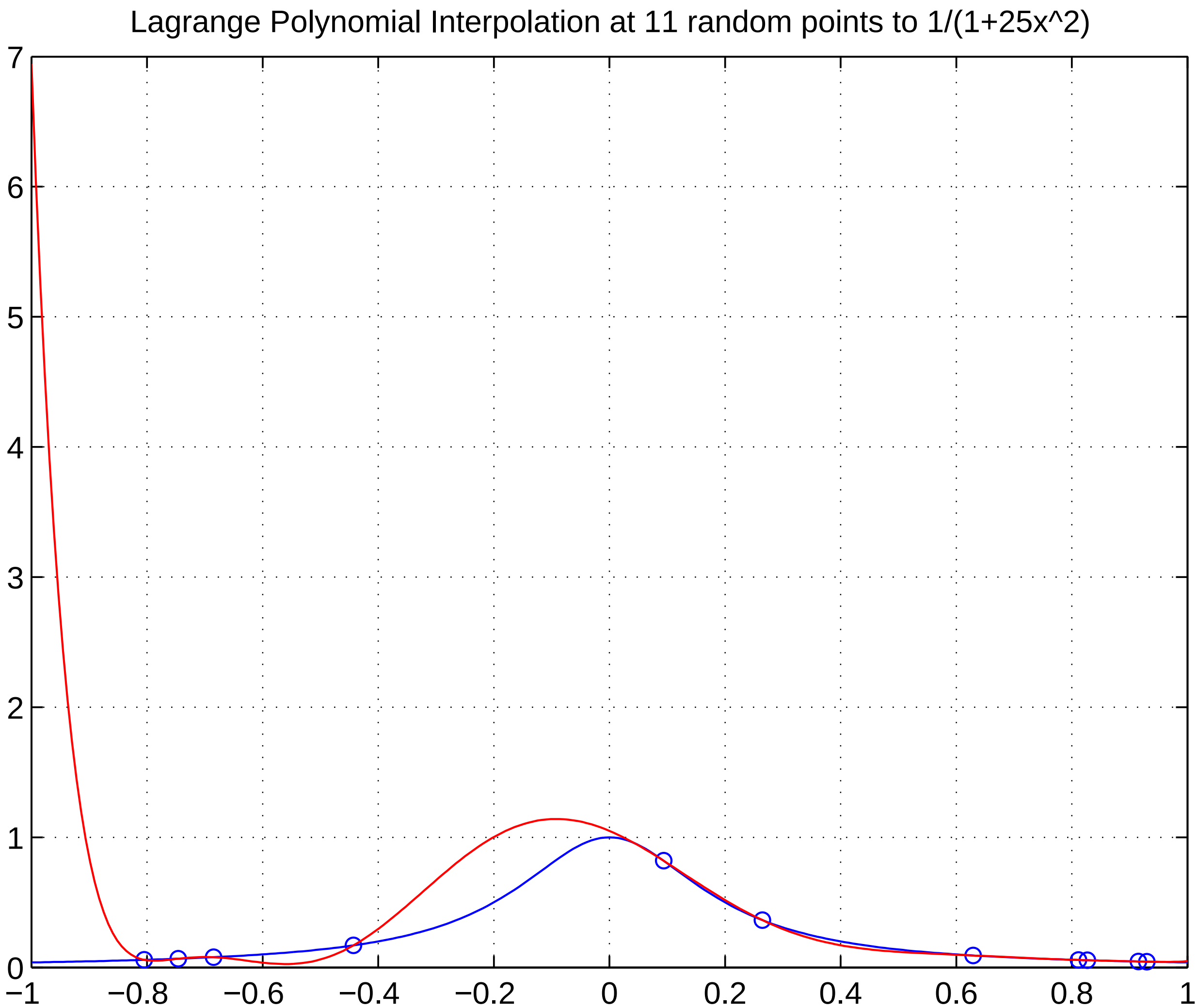
Carl Runge (1856–1927) reported the following observation: Consider $f(x) = \dfrac{1}{1+25x^2}$ over $[-1, 1]$, and an interpolation poly. $L_k$ at the equidistant nodes $x_j = -1 + j\frac{2}{k}$, $j = 0, 1, \cdots, k$.

Then, the interpolating poly. $L_k$ oscillates toward the edge of the interval $[-1, 1]$.

Moreover, $\lim_{k\to\infty} \left( \max_{-1 \le x \le 1} |f(x) - L_k(x)| \right) = +\infty$ !



Lagrange Polynomial Interpolation at 11 equispaced points to 1/(1+25x^2)
2
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1



Lagrange Polynomial Interpolation at 11 random points to 1/(1+25x^2)
7
6
5
4
3
2
1
0
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

(Proof) First, I leave the following as an exercise:

$$f(x) - L_k(x) = \frac{f^{(k+1)}(\xi)}{(k+1)!} \underbrace{\prod_{j=0}^{k}(x-x_j)}_{=\Phi(x)}, \quad \exists \xi \in [-1,1].$$

Then

$$\max_{x\in[-1,1]} |f(x)-L_k(x)| \le \max_{x\in[-1,1]} \frac{|f^{(k+1)}(x)|}{(k+1)!} \cdot \max_{x\in[-1,1]} \prod_{j=0}^{k} |x-x_j| \longrightarrow +\infty \text{ as } k\to\infty$$

again exercises. (see, e.g., Wikipedia on Runge's phenomenon.) //

$\Rightarrow$ Avoid global polynomial interpolation (at least for equidistant pts)!!

- Hence, the idea of using piecewise algebraic polynomials was conceived.
- We'll defer interpolation using trigonometric poly. until we learn Fourier series.
- Here, our discussion is based on the paper by Micula (2002).

<u>Notation</u>: $I := [a,b] \subset \mathbb{R}$

$\mathcal{P}_m := \{$ a set of poly's of deg. $\le m \}$

$H^m(I) := \{ f : I \to \mathbb{R} \mid f^{(m-1)} \in AC(I) \ \& \ f^{(m)} \in L^2(I) \}$

$= L^2$-Sobolev space $W^{m,2}(I)$

$AC(I)$: the space of abs. cont. fcns on $I$ $\Longleftrightarrow$ $f^{(m)}$ exists a.e. on $I$

<u>Def</u>. Let $\mathcal{P} := \{ a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b \}$ be a partition of $I$. Then $s : I \to \mathbb{R}$ is a polynomial spline of deg. $m$ w.r.t. $\mathcal{P}$

if $s \in C^{m-1}(I)$ & $s|_{[x_j, x_{j+1}]} \in \mathcal{P}_m$, $j = 0:n$.

The interior pts $\{x_1, \cdots, x_n\}$ are called knots.

## ☆ Natural Cubic Splines

Suppose that the knots and the values at the knots $\{y_1, \cdots, y_n\}$ are given.

Find a nice fcn $\varphi : I \to \mathbb{R}$ s.t. $\varphi(x_j) = y_j$, $j = 1:n$.

**Thm (Holladay 1957)**

Given $\{(x_j, y_j)\}_{j=1:n}$, $n \geq 2$,

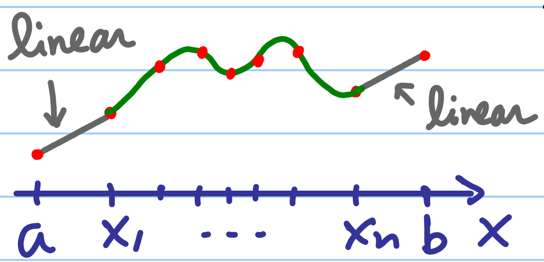
$\exists! \ \sigma \in X_n(I) := \{ f \in H^2(I) \mid f(x_j) = y_j, j = 1:n \}$

with $\int_I |\sigma''(x)|^2 dx = \min_{f \in X_n(I)} \int_I |f''(x)|^2 dx$

i.e., $\sigma = \arg\min_{f \in X_n(I)} \int_I |f''(x)|^2 dx$.

Furthermore,

$$\begin{cases} \bullet\ \sigma \in C^2(I) \\ \bullet\ \sigma|_{[x_j, x_{j+1}]} \in \mathcal{P}_3, \quad j = 1:n-1 \\ \bullet\ \sigma|_{[a, x_1]},\ \sigma|_{[x_n, b]} \in \mathcal{P}_1 \end{cases}$$




"Natural" since outside of $[x_1, x_n]$, $\sigma$ is linear.

why would one choose to minimize $\int_I |f''(x)|^2 dx$?

(A1) The curvature of $f$ is $\frac{f''}{(1+f'^2)^{3/2}}$, which implies

$\sigma \approx$ minimizes the total curvature if $\sigma'$ is small!

i.e., $\sigma$: smooth, no kinks.

(**Note:** Why the curvature of $f$ is $\frac{f''}{(1+f'^2)^{3/2}}$ ?

**Ans:** For a plane curve $s(t) = (x(t), y(t))$, its curvature is $\frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{3/2}}$, which is well-known. Then set $x(t) = t$ to get $\ddot{y}/(1+\dot{y}^2)^{3/2}$)

(A2) Nat. Cubic Spline $\approx$ the solution of a physical problem of deformation of a bar of uniform, thin, elastic material without external force.

$\Rightarrow$ min. elastic potential energy

$\approx \min \int_I |f''(x)|^2 dx$

(A3) $\sigma$ can be interpreted as a curve closest to a straight line yet passing through those given pts.

**Note:** For each subinterval, if we want

$$I_j = \int_{x_{j-1}}^{x_j} f(x, y, y', y'')\, dx \longrightarrow \min!$$

then the E-L eqn. becomes

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} = 0$$

Deriving this is an excercise!

Now let $f = y''^2$. Then the E-L eqn. becomes $y^{(4)} = 0$ with an appropriate B.C.

$\longrightarrow$ Biharmonic eqn. (1D)

⇒ All sorts of generalization have been developed. See Micula and the references therein.

⇒ For computational algorithm, see spline fcn in MATLAB, and Algorithm 472 (ACM) by Herriot & Reinsch.

(Proof of the Holladay Thm)

It suffices to prove for any $f \in X_n(I)$,

$$\int_I |f''(x)|^2 dx > \int_I |\sigma''(x)|^2 dx.$$

Let $\eta(x) := f(x) - \sigma(x)$, i.e., $f(x) = \eta(x) + \sigma(x)$

Since $\sigma''' \equiv$ const. $=: c_j$ in each $[x_j, x_{j+1}]$, we have

$$\int_I |f''(x)|^2 dx - \int_I |\sigma''(x)|^2 dx$$

$$= \int_I |\eta''(x)|^2 dx + 2\int_I \eta''(x)\sigma''(x)\, dx$$

$$= \int_I |\eta''(x)|^2 dx + 2\sum_{j=0}^{n} \int_{x_j}^{x_{j+1}} \eta''(x)\sigma''(x)\, dx$$

$$= \int_I |\eta''(x)|^2 dx + 2\sum_{j=0}^{n} \Big[ \underbrace{\eta'\sigma''\Big|_{x_j}^{x_{j+1}}}_{\text{telescopic sum!}} - \int_{x_j}^{x_{j+1}} \eta' \underbrace{\sigma'''}_{= c_j} dx \Big]$$

$$= \int_I |\eta''(x)|^2 dx + 2\left( \eta'(b)\sigma''(b) - \eta'(a)\sigma''(a) \right)$$

$$- 2\sum c_j \underbrace{\left(\eta(x_{j+1}) - \eta(x_j)\right)}_{= 0 \text{ since } f(x_j) = \sigma(x_j)}$$

$$= \int_I |\eta''(x)|^2 dx \text{ by setting } \underbrace{\sigma''(a) = \sigma''(b) = 0}_{\text{Hence, } \sigma:\text{ linear in } [a, x_1], [x_n, b].}$$

$$> 0 \quad //$$