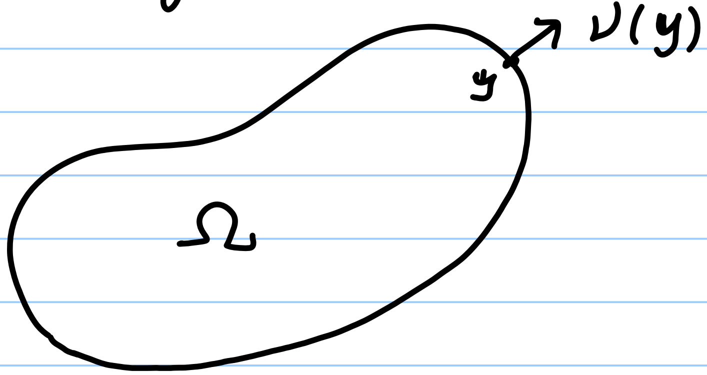
Lecture 10: Basics of PDEs IV

Note Title

☆ Basic Properties of Harmonic Functions

Def. A C^2 fcn u defined on $\Omega \subset \mathbb{R}^d$ is said to be harmonic if $\Delta u = 0$.

Here, Ω is a domain, i.e., an open subset of $\mathbb{R}^d$ (not necessarily connected). Its boundary is denoted by $\partial\Omega$.
Let $d\sigma$ be the surface measure on $\partial\Omega$. ν be the unit normal vector on $\partial\Omega$ pointing out of Ω.



The normal derivative of f differentiable near $\partial\Omega$ is defined by

$$\frac{\partial f}{\partial \nu}(y) = \partial_\nu f(y) := \nu(y)\cdot\nabla f(y)$$

* Green's Identities

If Ω is a bdd. domain with smooth bdy, and $u, v \in C^2(\overline{\Omega})$, then

1st id: $$\int_{\partial\Omega} v\,\partial_\nu u\, d\sigma = \int_\Omega (v\Delta u + \nabla v\cdot\nabla u)\,dx$$

2nd id: $$\int_{\partial\Omega} (v\,\partial_\nu u - u\,\partial_\nu v)\,d\sigma = \int_\Omega (v\Delta u - u\Delta v)\,dx$$

Remark: These are $\mathbb{R}^d$ versions of the integration by parts!!
In $\mathbb{R}^1$, e.g., 1st id: $v u'\big|_a^b = \int_a^b (v u'' + v'u')\,dx$

$$\iint_S \mathbb{F}\cdot\mathbf{n}\, dS = \iiint_V \nabla\cdot\mathbb{F}\, dV$$

(Proof) The 1st identity is just the Divergence Thm applied to the vector field $v\nabla u$.
The 2nd identity follows easily by swapping u & v and then subtracting. ///

Cor: If u is harmonic on Ω, then

$$\int_{\partial\Omega} \partial_\nu u \, d\sigma = 0.$$

(Proof) Easy! Take $v \equiv 1$ in the 1st id. ///

* **The Mean Value Thm**

Suppose u is harmonic on $\Omega \subset \mathbb{R}^d$.
If $x \in \Omega$, $r > 0$ is small s.t. $\overline{B^d(x;r)} \subset \Omega$, then

closure

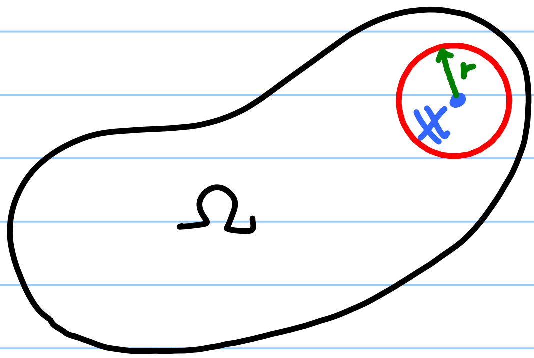
an open ball in $\mathbb{R}^d$ with x center r radius

$$u(x) = \frac{1}{r^{d-1}\omega_d}\int_{S^{d-1}(x;r)} u(y)\, d\sigma(y)$$
$$= \frac{1}{\omega_d}\int_{S^{d-1}(0;1)} u(x+ry)\, d\sigma(y),$$

where $\omega_d := 2\pi^{d/2}/\Gamma(d/2)$ = the surface area of $S^{d-1}(x;1)$

the unit sphere in $\mathbb{R}^d$

A Good Exercise to derive this!



(Proof) See, e.g., G.B. Folland: Intro to PDE, 2nd Ed. ///

* The converse to the MV Thm

Suppose $u \in C(\Omega)$, and $\forall x \in \Omega$ s.t. $\overline{B^d(x;r)} \subset \Omega$

and $u(x) = \frac{1}{\omega_d} \int_{S^{d-1}(0;1)} u(x + r y)\, d\sigma(y)$.

Then $u \in C^\infty(\Omega)$ and u is harmonic on Ω.

(Proof) See Folland. ///

Applying both the M.V. Thm & its converse, we get the following

Cor: u: harmonic on $\Omega \Rightarrow u \in C^\infty(\Omega)$.

* The Maximum Principle

Suppose Ω is a connected domain.

If u is harmonic and real-valued on Ω and $\sup_{x \in \Omega} u(x) = A < \infty$, then

either $u(x) < A$, $\forall x \in \Omega$

or $u(x) \equiv A$, $\forall x \in \Omega$.

(Proof) The set $\Omega_A := \{x \in \Omega \mid u(x) = A\}$ is relatively closed in Ω (i.e., $\exists$ a closed subset $K \subset \mathbb{R}^d$ s.t. $\Omega_A = K \cap \Omega$).

Now by the M.V. Thm, if $u(x) = A$, then $u(y) = A$, $\forall y \in B^d(x;r)$, $\forall r > 0$ with $\overline{B^d(x;r)} \subset \Omega$.

(if not, the m.v. on $S^{d-1}(x;r)$ would be $< A$ because $A = \sup_{x \in \Omega} u(x)$).

$\Rightarrow \Omega_A$ is also open.

$\Rightarrow \Omega_A = \phi$ or Ω. ///

Cor: Suppose $\bar{\Omega}$ is compact.
If u is harmonic and real-valued on Ω and $u \in C(\bar{\Omega})$, **then** the max. of u on $\bar{\Omega}$ is achieved on $\partial\Omega$.

(Proof) Since $u \in C(\bar{\Omega})$, the max. is achieved somewhere $\mathbf{x}^* \in \bar{\Omega}$. If $\mathbf{x}^* \in \Omega$, then $u = \text{const.}$ on the connected component of Ω containing $\mathbf{x}^*$ so the max. is also achieved on the boundary of that component. ///

* **The Uniqueness Thm**

Suppose $\bar{\Omega}$ is compact. **If** u_1, u_2 are harmonic fcns on Ω, in $C(\bar{\Omega})$, and $u_1 = u_2$ on $\partial\Omega$, **then** $u_1 \equiv u_2$ on Ω.

(Proof) $\mathrm{Re}(u_1 - u_2)$ & $\mathrm{Im}(u_1 - u_2)$ are harmonic on Ω. $\Rightarrow$ They achieve their max. on $\partial\Omega$, which is 0 since $u_1 = u_2$ on $\partial\Omega$.
By the maximum principle, either
$u_1 - u_2 < 0$ or $u_1 - u_2 = 0$ on Ω.
Do the same for $u_2 - u_1$ to conclude
$u_1 \equiv u_2$ on Ω. ///

* **Liouville's Thm**

If u is bdd. and harmonic on $\mathbb{R}^d$, **then** $u \equiv \text{const.}$

(Proof) See Folland.

☆ A Fundamental Solution

Def. Let $\mathcal{L}$ = a const. coefficient linear partial differential operator. Then the distribution Φ s.t. $\mathcal{L}\Phi = \delta$ (the Dirac delta) is called a fundamental solution for $\mathcal{L}$ (a.k.a. a free space Green's fcn for $\mathcal{L}$).

So, if we have a problem $\mathcal{L}u = f$ in $\mathbb{R}^d$ then take $u = \Phi * f$. (convolution)

$$\Rightarrow \mathcal{L}u = \mathcal{L}(\Phi * f) = (\mathcal{L}\Phi) * f = \delta * f = f \checkmark$$

If $\mathcal{L} = -\Delta$ (negative Laplacian), then we have

$$\Phi(x,y) = \begin{cases} -\frac{1}{2}|x-y| & \text{if } d=1 \\ -\frac{1}{2\pi}\ln|x-y| & \text{if } d=2 \\ \frac{|x-y|^{2-d}}{(d-2)\omega_d} & \text{if } d \geq 3 \end{cases}$$

Note that $d=3$ leads to the famous $\frac{1}{4\pi|x-y|}$.