Lecture 11 : Basics of PDEs V

Note Title

☆ The Dirichlet & Neumann Problems

Given fcns f on Ω, g on $\partial\Omega$,
find a fcn u on $\overline{\Omega}$ s.t.

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \Rightarrow \text{the Dirichlet problem}$$

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \partial_\nu u = g & \text{on } \partial\Omega \end{cases} \Rightarrow \text{the Neumann problem}$$

Here Ω is a bdd. domain, not necessarily connected.

* For the Dirichlet problem, if the sol. exists, then it's unique as we already showed.

* For the Neumman problem, the uniqueness doesn't hold! why?
  $\Rightarrow$ u + any const. is also a sol.
  Also, $\exists$ a necessary solvability cond. for the Neumann problem as follows.

Let Ω' be any connected component of Ω.
Then, using Green's 2nd id.

$$\int_{\Omega'} (v\Delta u - u\Delta v)\, dx = \int_{\partial\Omega'} (v\partial_\nu u - u\partial_\nu v)\, d\sigma$$

and setting $v \equiv 1$, we have

$$\int_{\Omega'} \underbrace{\Delta u}_{=f} \, dx = \int_{\partial\Omega'} \underbrace{\partial_\nu u}_{=g} \, d\sigma$$

So, f & g must satisfy

$$\int_{\Omega'} f \, dx = \int_{\partial\Omega'} g \, d\sigma$$ on each connected component of Ω.

In particular, in the case of Laplace's eqn. (i.e., $f \equiv 0$), g must satisfy

$$\int_{\partial\Omega'} g \, d\sigma = 0$$ on each connected component of Ω.

* The reduction of the Dirichlet problem to $f = 0$ or $g = 0$.

If we can find fcns v & w s.t.

① $\begin{cases} \Delta v = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$ Poisson with homog. Dirichlet B.C.

② $\begin{cases} \Delta w = 0 & \text{in } \Omega \\ w = g & \text{on } \partial\Omega, \end{cases}$ Laplace with inhomog. Dirichlet B.C.

then clearly $u = v + w$ satisfies the original Dirichlet problem!

* Equivalence of ① & ②

Now, we will demonstrate that ① & ② are more or less equivalent. This is important since a numerical solver developed for ① can also handle and solve ② !!

Suppose we can solve ① and wish to solve ②.
assume that g has an extension $\tilde{g} \in C^2(\overline{\Omega})$
s.t. $\tilde{g} = g$ on $\partial\Omega$.
Then, we can find v satisfying

$$\begin{cases} \Delta v = \Delta \tilde{g} & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

$\Rightarrow$ This is a form of ①
so we can solve this.

Now take $w = \tilde{g} - v$.

$$\Rightarrow \quad \Delta w = \Delta \tilde{g} - \Delta v = 0 \quad \text{in } \Omega$$

$$w|_{\partial\Omega} = \tilde{g}|_{\partial\Omega} - v|_{\partial\Omega} = g - 0 = g$$

So, ② is solved! ✓

On the other hand, suppose we can solve ② and wish to solve ①.
First, let's extend f to the outside of Ω by zero, i.e., $\tilde{f} = \begin{cases} f & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^d \setminus \Omega \end{cases}$
and set $v' = \tilde{f} * \Phi$ so that $\Delta v' = \tilde{f}$
Second, we solve $\begin{cases} \Delta w = 0 & \text{in } \Omega \\ w = v' & \text{on } \partial\Omega. \end{cases}$ $\Rightarrow$ This is of ②

Now take $v = v' - w$.
Then we see

$$\begin{cases} \Delta v = \Delta v' - \Delta w = f - 0 = f & \text{in } \Omega \\ v|_{\partial\Omega} = v'|_{\partial\Omega} - w|_{\partial\Omega} = v'|_{\partial\Omega} - v'|_{\partial\Omega} = 0. \end{cases}$$

So, ① is solved!! ///

<u>Exercise</u>: Prove the Neumann case.

☆ A Fast & Accurate Laplace/Poisson Solver on a Rectangle

(A. Averbuch, M. Israeli, and L. Vozovoi, SIAM J. Sci. Comput., vol. 19, no. 3, pp. 933-952, 1998)

Consider, for simplicity, $\Omega = (0,1)\times(0,1) \subset \mathbb{R}^2$.

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases}$$

Then AIV showed that the sol. to this Dirichlet problem as

$$u(x,y) = p(x,y) + \sum_{k\ge 1} \left\{ b_k^{(1)} h_k(x, 1-y) + b_k^{(2)} h_k(y, 1-x) + b_k^{(3)} h_k(x,y) + b_k^{(4)} h_k(y,x) \right\}$$

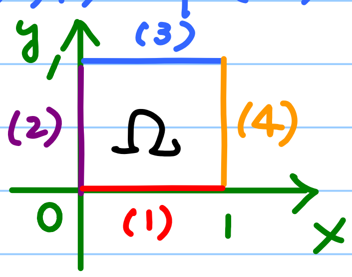
$p(x,y) \rightarrow$ is harmonic!

where $p(x,y) = a_3 xy + a_2 x + a_1 y + a_0$

s.t. $p(x,y) = f(x,y)$ at the 4 corners $(x,y) = (0,0), (1,0), (0,1), (1,1)$,

$$h_k(x,y) := \sin(\pi k x)\,\frac{\sinh(\pi k y)}{\sinh(\pi k)},$$

and $\{b_k^{(1)}\}, \{b_k^{(2)}\}, \{b_k^{(3)}\}, \{b_k^{(4)}\}$ are the Fourier sine series coeff's of $f(x,0)-p(x,0)$, $f(0,y)-p(0,y)$, $f(x,1)-p(x,1)$, $f(1,y)-p(1,y)$, respectively.



i.e.,
$$\sum b_k^{(1)} \sin \pi k x = f(x,0) - p(x,0)$$
$$\sum b_k^{(2)} \sin \pi k y = f(0,y) - p(0,y)$$
$$\sum b_k^{(3)} \sin \pi k x = f(x,1) - p(x,1)$$
$$\sum b_k^{(4)} \sin \pi k y = f(1,y) - p(1,y)$$

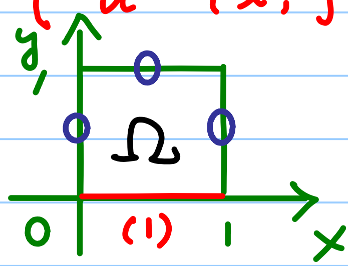
In practice, one needs to truncate these Fourier sine series. Also x, y must be discretized, then FFT-based Discrete Sine Transform should be used.

Note that

$$u^{(1)}(x,y) := \sum_{k\geq 1} b_k^{(1)} h_k(x, 1-y)$$

is the sol. to the following Dirichlet problem:

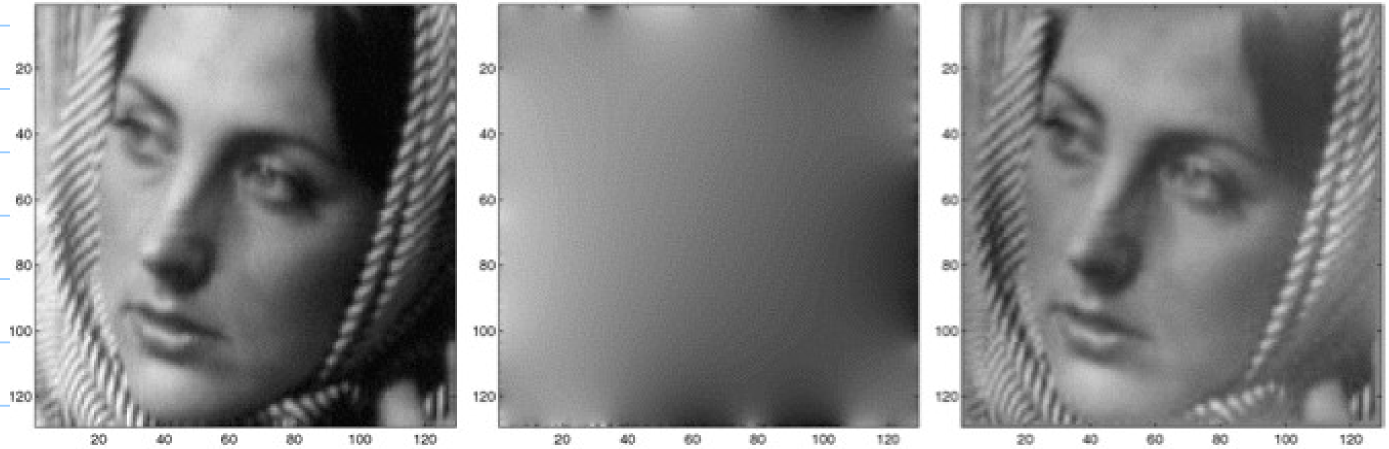
$$\begin{cases} \Delta u^{(1)} = 0 & \text{in } \Omega \\ u^{(1)}(x,0) = f(x,0) - p(x,0) & \text{on } (1) \subset \partial\Omega \\ u^{(1)}(x,y) = 0 & \text{on } (2),(3),(4) \subset \partial\Omega \end{cases}$$



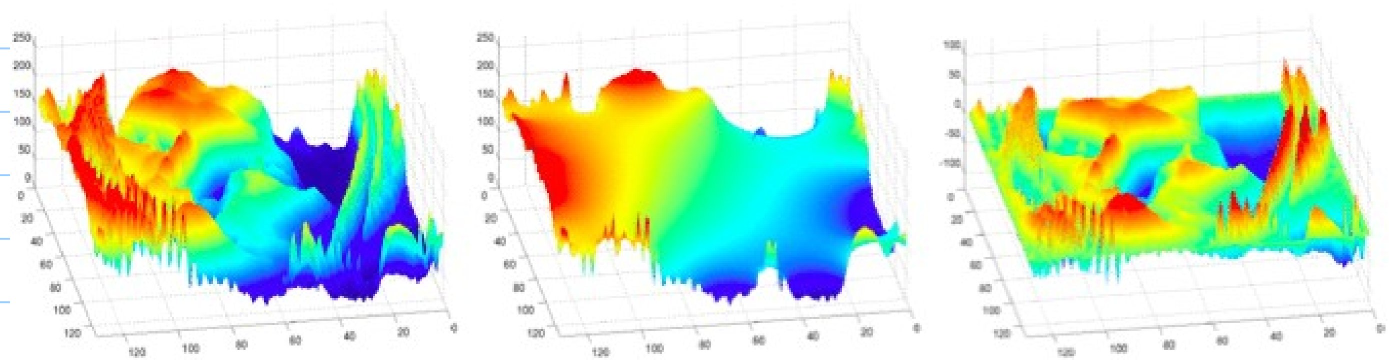
So as the other three sums.

$$\Rightarrow u(x,y) = p(x,y) + \boxed{u^{(1)}} + \boxed{u^{(2)}} + \boxed{u^{(3)}} + \boxed{u^{(4)}}$$

$\Rightarrow$ No finite difference approx. of Δ !



(a) Original (b) u component (c) v component



(d) Original (e) u component (f) v component

u is the solution of the Dirichlet problem using the AIV method.

$v = f - u$.

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

where f is the original image intensities in (a), (d).