Lecture 12: Fourier Series I

Note Title

☆ Fourier Series of a Periodic Function

Def. Suppose f is defined on the real axis s.t., $f(\theta+2\pi) = f(\theta)$, $\forall \theta \in \mathbb{R}$.
Such f is called 2π-periodic.

We assume f: Riemann integrable on every bdd. interval, and $\mathbb{C}$-valued.
Want to know if f can be expanded in a series:

$$(*) \quad f(\theta) = \frac{1}{2}\underbrace{a_0}_{\text{corresp. to } n=0 \text{ case}} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

for convenience

We can express $(*)$ as

$$(**) \quad f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

thanks to $\cos n\theta = \dfrac{e^{in\theta}+e^{-in\theta}}{2}$, $\sin n\theta = \dfrac{e^{in\theta}-e^{-in\theta}}{2i}$

$\Rightarrow$ an important relationship:

$$a_0 = 2c_0, \quad a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n}), \quad n \in \mathbb{N}.$$

Now, how can we compute c_n (or a_n & b_n)?

Formally,

$$f(\theta)\cdot e^{-ik\theta} = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}\cdot e^{-ik\theta}$$

$$\Rightarrow \int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}\,d\theta = \int_{-\pi}^{\pi}\sum_{-\infty}^{\infty} c_n e^{i(n-k)\theta}\,d\theta$$

Need justification

$$= \sum_{-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-k)\theta}\,d\theta \quad \text{——} \quad (☆)$$

It's an easy exercise to derive:

$$\int_{-\pi}^{\pi} e^{i(n-k)\theta}\,d\theta = 2\pi\underbrace{\delta_{n,k}}_{\text{the Kronecker delta}} = \begin{cases} 2\pi & \text{if } n=k \\ 0 & \text{if } n\neq k. \end{cases}$$

So, $(\star) = \sum_{-\infty}^{\infty} c_n\, 2\pi\,\delta_{n,k} = 2\pi c_k$, i.e.,

$$(\star\star)\begin{cases} c_n = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta, \quad n\in\mathbb{Z} \\ \text{Using the } c_n \longleftrightarrow (a_n, b_n) \text{ formulas, we get} \\ a_0 = \underset{\uparrow\, n=0}{2c_0} = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta)\,d\theta \to \text{often called the DC component or mean val.} \\ a_n = c_n + c_{-n} = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta \\ b_n = i(c_n - c_{-n}) = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta \end{cases}, \quad n\in\mathbb{N}.$$

So, <u>if</u> the series converges nicely so that term-by-term integrals in $(\star)$ are justified, <u>then</u> the coeff's are given as $(\star\star)$.
But, <u>if</u> f: Riemann integrable & periodic, <u>then</u> the RHS of $(\star\star)$ make perfectly good sense and we can define these coeff's.

<u>Def</u>. Suppose f is 2π-periodic & Riemann-integrable on $[-\pi,\pi]$. Then the numbers c_n, a_n, b_n defined by $(\star\star)$ are called the **Fourier coef's** of f, and the corresponding series

$$\sum_{-\infty}^{\infty} c_n e^{in\theta}, \quad \frac{1}{2}a_0 + \sum_{1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta)$$

are called the **Fourier series** of f.

Remark: Thanks to the 2π-periodicity of f, we can replace $\int_{-\pi}^{\pi} \cdot \, d\theta$ by $\int_{a}^{2\pi+a} \cdot \, d\theta$, any $a \in \mathbb{R}$, e.g., $a=0$.

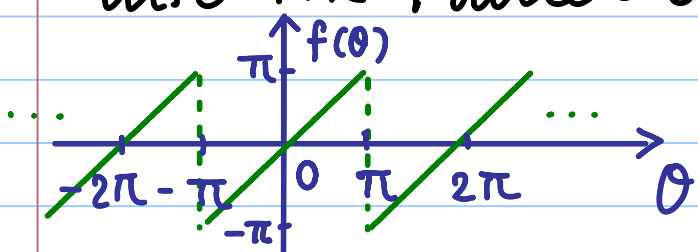
Convenient Facts:

<u>If</u> F is even, i.e., $F(-x) = F(x)$
<u>then</u> $\int_{-a}^{a} F(x)\,dx = 2\int_{0}^{a} F(x)\,dx$.

<u>If</u> F is odd, i.e., $F(-x) = -F(x)$
<u>then</u> $\int_{-a}^{a} F(x)\,dx = 0$.

Hence, f: even $\Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta\, d\theta$, $b_n \equiv 0$.
f: odd $\Rightarrow a_n \equiv 0$, $b_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta\, d\theta$.

Example 1. Expand $f(\theta) = \theta$, $-\pi \le \theta \le \pi$ into the Fourier Series.



This is an odd fcn.
$\Rightarrow$ Expand with only sine terms.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \theta \sin n\theta \, d\theta = \frac{2}{\pi}\int_0^{\pi} \theta \sin n\theta \, d\theta, \quad n \ge 1.$$

Int. by Parts

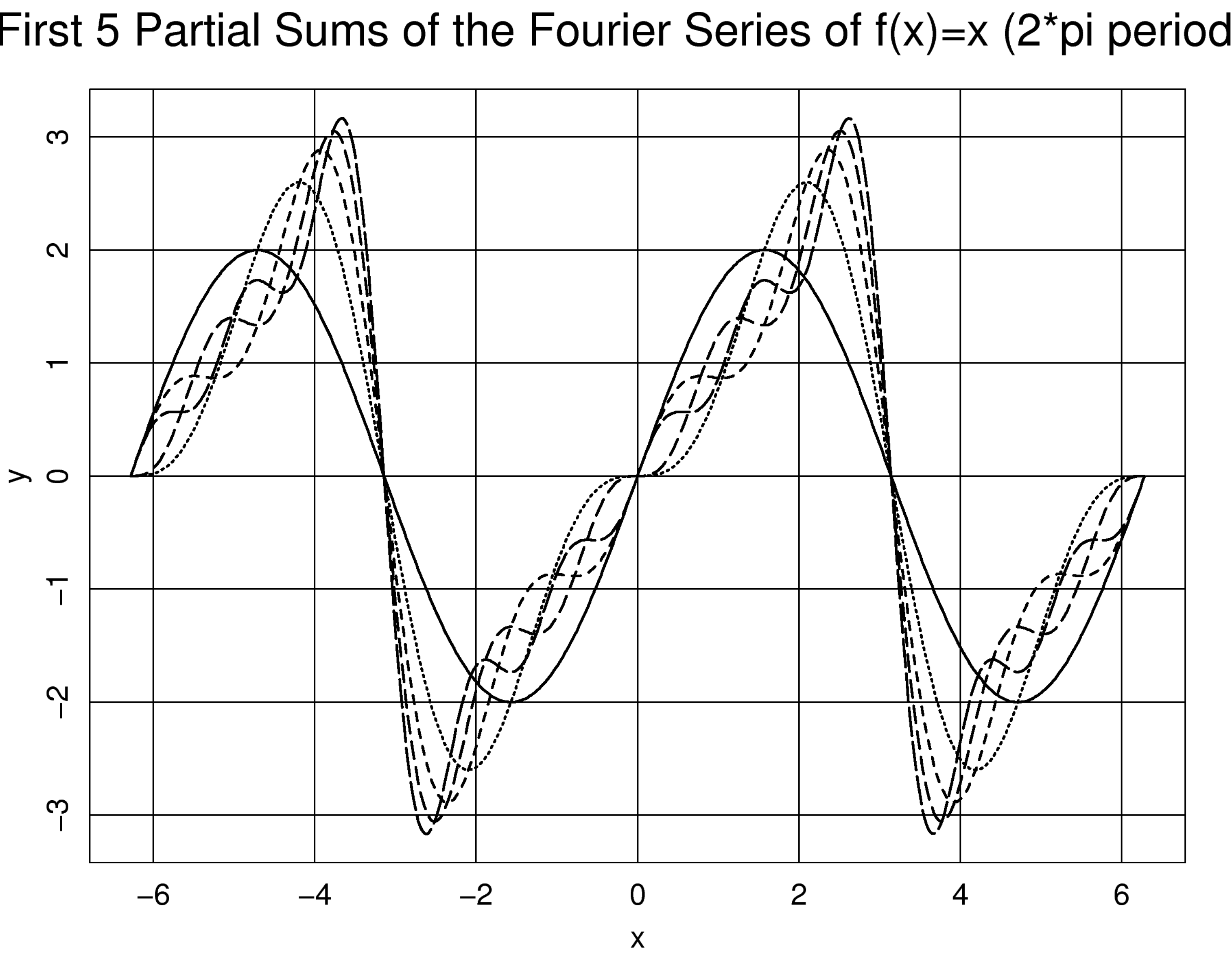
$$\overset{\downarrow}{=} \frac{2}{\pi}\left\{ -\frac{\cos n\theta}{n}\theta \Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi} \cos n\theta \cdot 1 \cdot d\theta \right\}$$

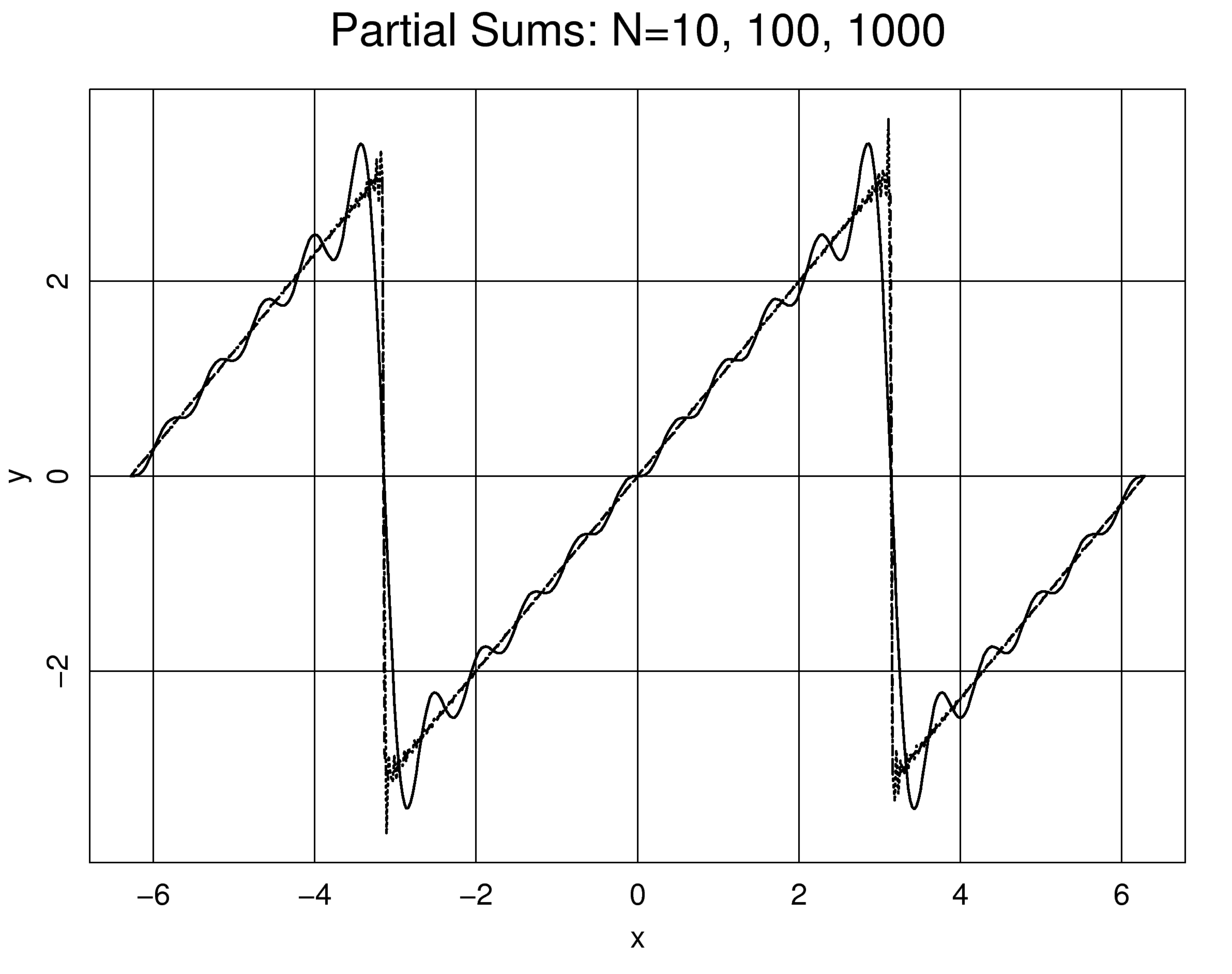
$$= \frac{2}{\pi}\left\{ -\frac{\pi \cos n\pi}{n} + \cancel{\frac{1}{n^2}\sin n\theta \Big|_0^{\pi}}_{\,0} \right\} \underset{\uparrow}{=} \frac{2}{n}(-1)^{n+1}$$

$\cos n\pi = (-1)^n$

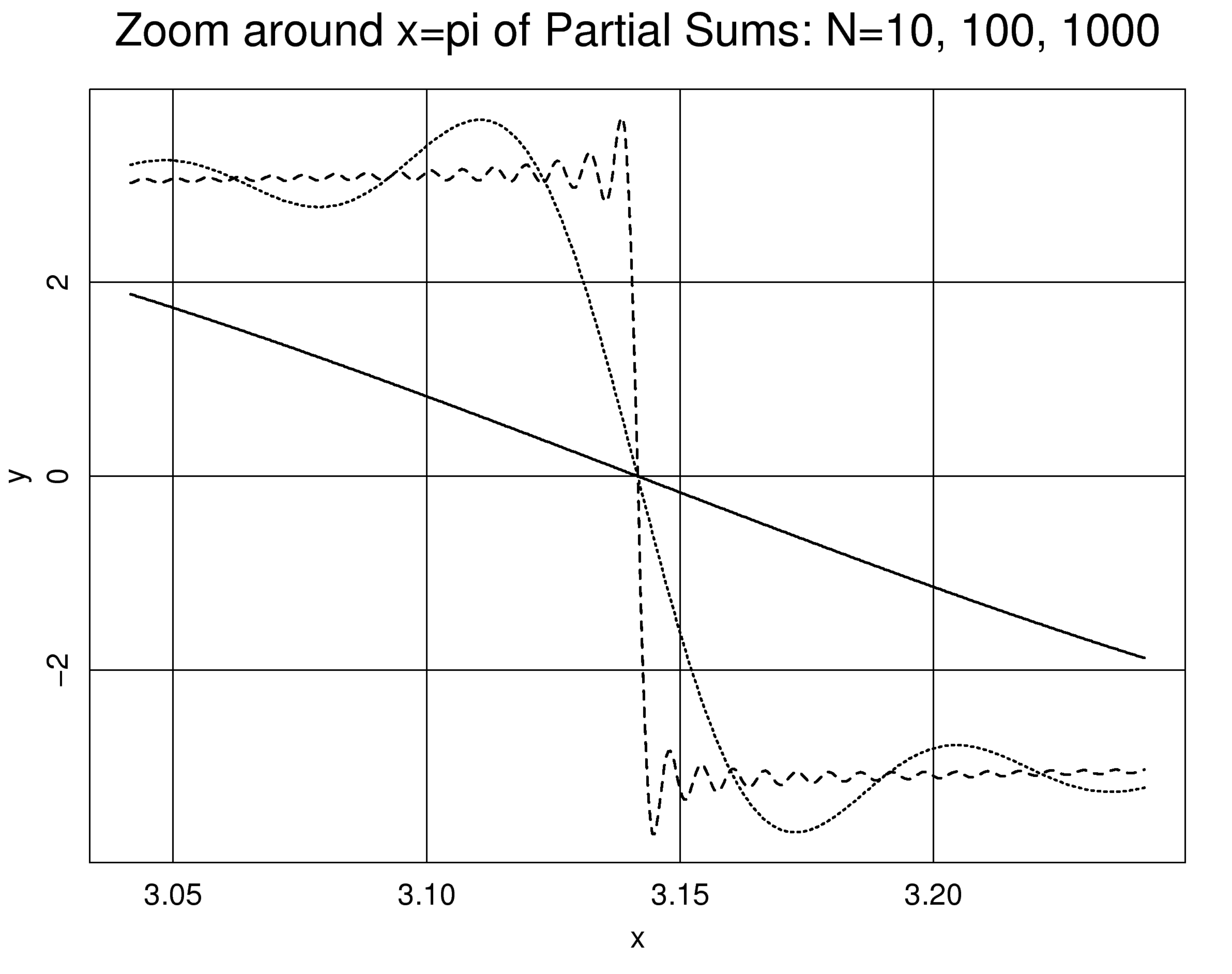
$$\Rightarrow f(\theta) = \theta \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin n\theta \quad //$$

Use $\sim$ since for $\theta = k\pi$, RHS $= 0$ while LHS is undefined.

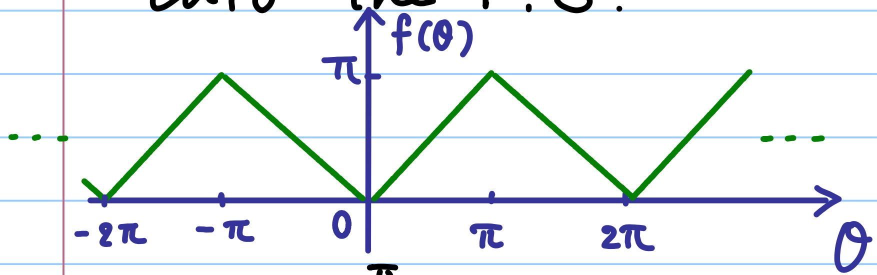

First 5 Partial Sums of the Fourier Series of f(x)=x (2*pi period
y
x
−6
−4
−2
0
2
4
6
−3
−2
−1
0
1
2
3



Partial Sums: N=10, 100, 1000
y
2
0
−2
−6
−4
−2
0
2
4
6
x


Zoom around x=pi of Partial Sums: N=10, 100, 1000
y
2
0
−2
3.05
3.10
3.15
3.20
x

Example 2. Expand $f(\theta) = |\theta|$, $-\pi \le \theta \le \pi$ into the F.S.



This is an even fcn.
$\Rightarrow$ Expand with only cosine terms.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\theta| \cos n\theta \, d\theta \qquad n \ge 1.$$

$$= \frac{2}{\pi}\int_0^{\pi} \theta \cos n\theta \, d\theta$$

$$= \frac{2}{\pi}\left\{ \cancelto{0}{\frac{\sin n\theta}{n}\theta\Big|_0^{\pi}} - \frac{1}{n}\int_0^{\pi} \sin n\theta \cdot 1 \, d\theta \right\}$$

$$= \frac{2}{\pi}\cdot\left\{ -\frac{1}{n}\cdot\frac{-\cos n\theta}{n}\Big|_0^{\pi} \right\} = \frac{2}{\pi}\cdot\frac{(-1)^n - 1}{n^2}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \theta \, d\theta = \frac{2}{\pi}\,\frac{\theta^2}{2}\Big|_0^{\pi} = \pi$$

$$\Rightarrow f(\theta) = |\theta| \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta$$

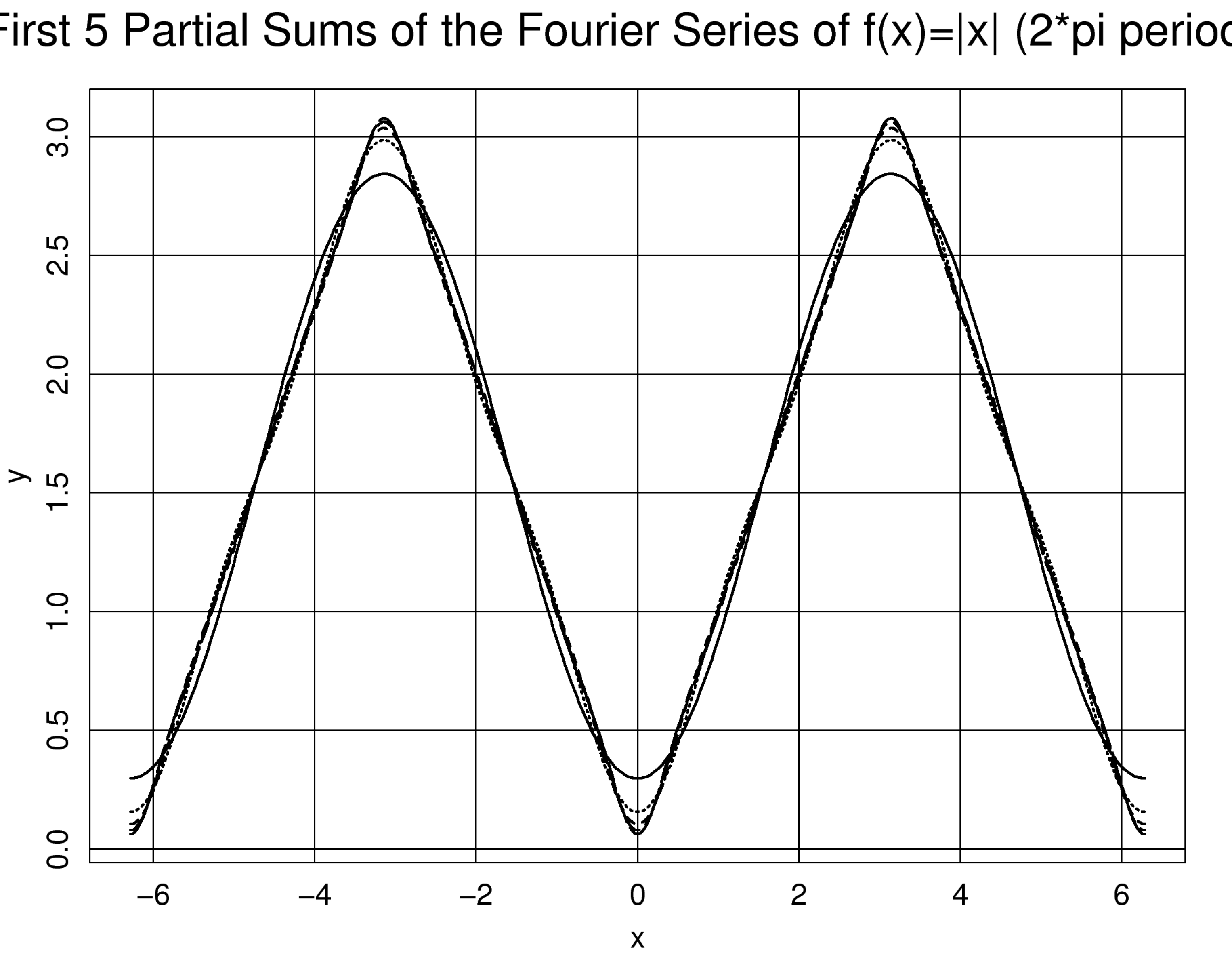
$$= \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2}\cos n\theta$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{-2}{(2m-1)^2}\cos(2m-1)\theta$$

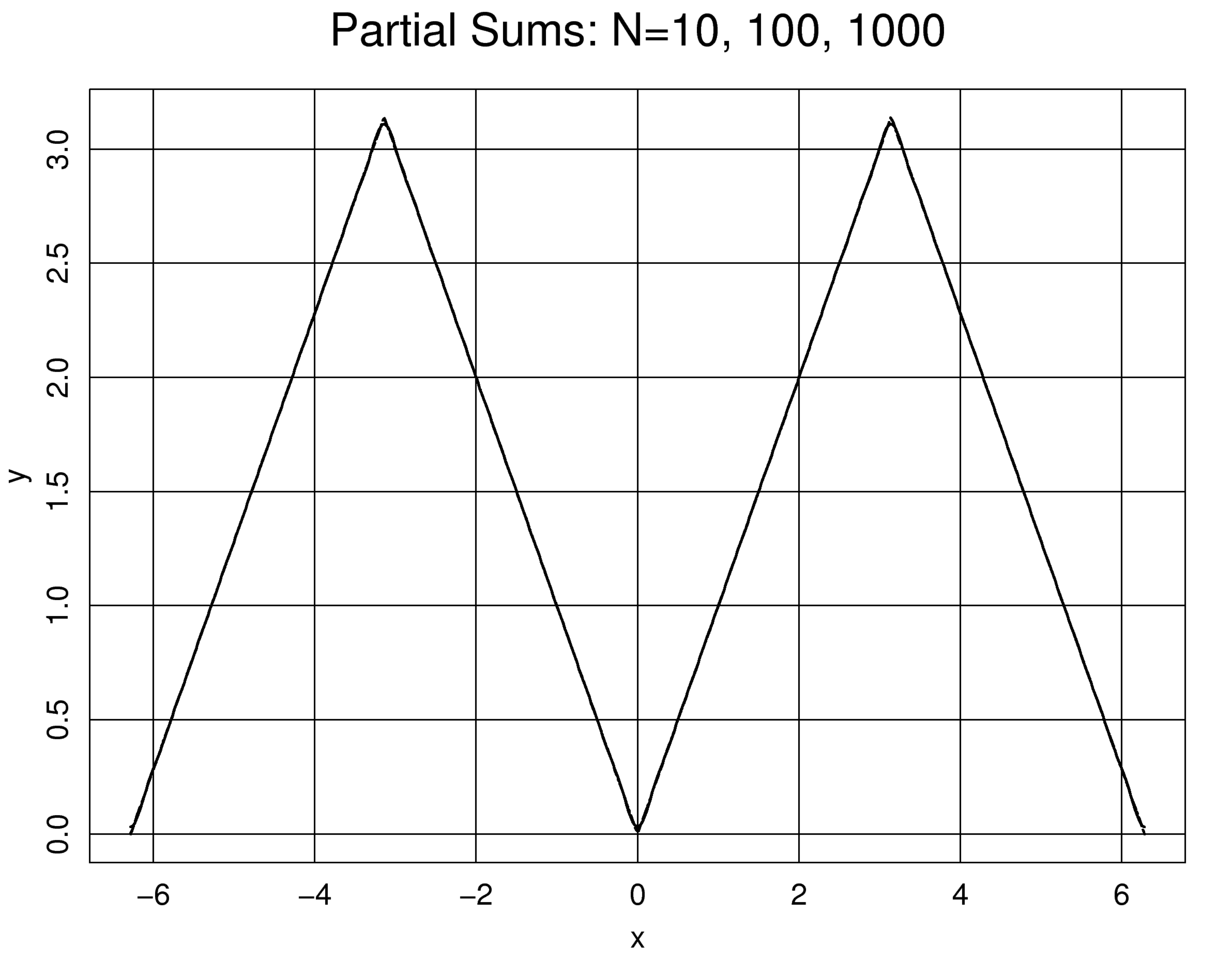
$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{\cos(2m-1)\theta}{(2m-1)^2} \quad //$$

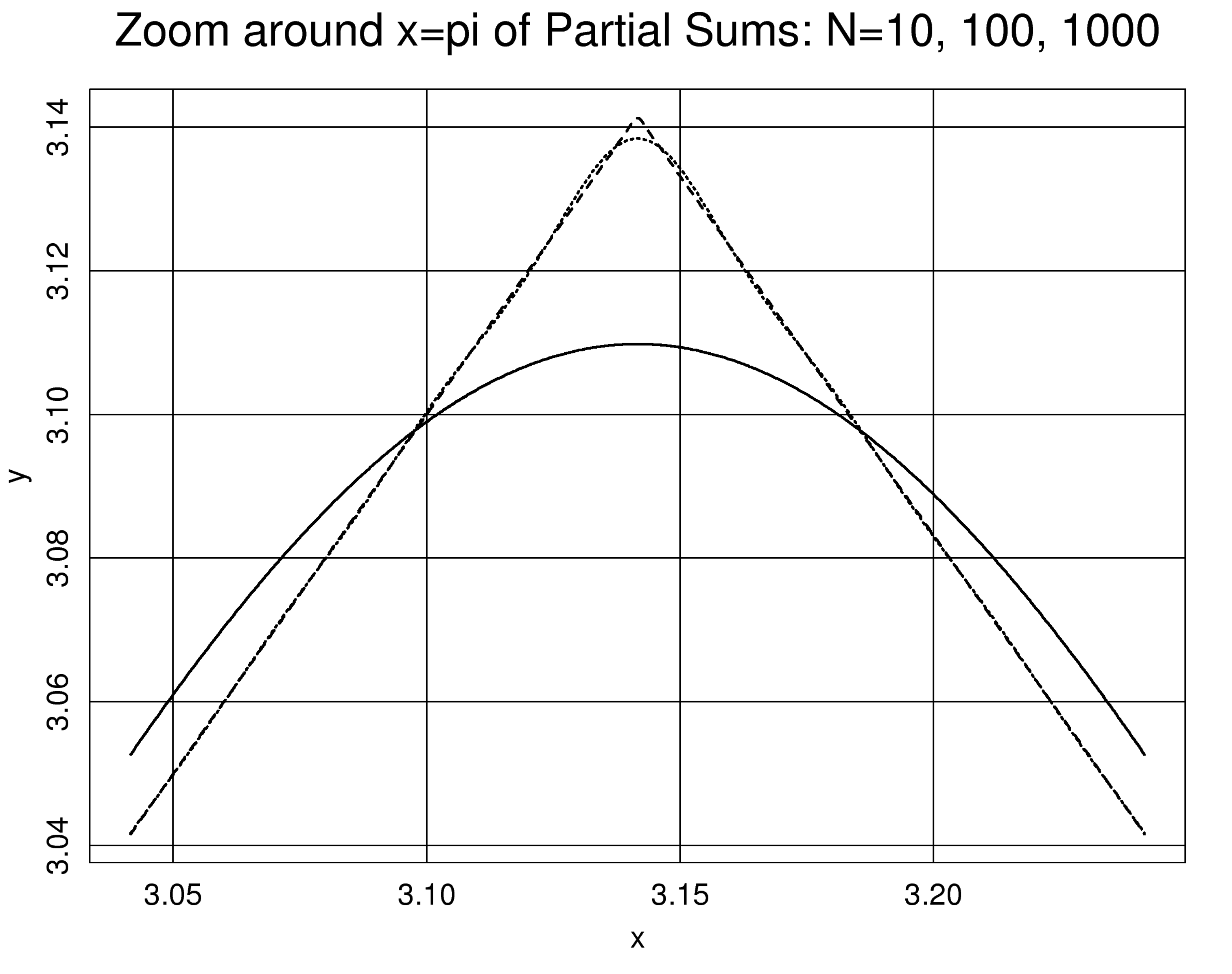
You can see that the convergence of this F.S. is faster than that of Example 1!

Exercise: Derive the same F.S. using the complex F.S. with c_n formula.



First 5 Partial Sums of the Fourier Series of f(x)=|x| (2*pi period
y
0.0
0.5
1.0
1.5
2.0
2.5
3.0
−6
−4
−2
0
2
4
6
x


Partial Sums: N=10, 100, 1000
y
0.0
0.5
1.0
1.5
2.0
2.5
3.0
x
−6
−4
−2
0
2
4
6


Zoom around x=pi of Partial Sums: N=10, 100, 1000
3.04
3.06
3.08
3.10
3.12
3.14
y
3.05
3.10
3.15
3.20
x

☆ Bessel's Inequality

Let f be 2π-periodic & Riemann integrable on $[-\pi,\pi]$.
Let c_n be the Fourier coef of f.
Then,
$$\sum_{-\infty}^{\infty} |c_n|^2 \le \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta}_{\text{mean energy}}$$

(Proof)
$$\Big| f(\theta) - \underbrace{\sum_{-N}^{N} c_n e^{in\theta}}_{N\text{-th partial sum}} \Big|^2$$
$$= \left(f(\theta) - \sum_{-N}^{N} c_n e^{in\theta}\right)\left(\overline{f(\theta)} - \sum_{-N}^{N} \bar{c}_n e^{-in\theta}\right)$$
$$= |f(\theta)|^2 - 2\,\mathrm{Re}\left\{f(\theta)\sum_{-N}^{N} \bar{c}_n e^{-in\theta}\right\} + \sum_{m,n=-N}^{N} c_m \bar{c}_n e^{i(m-n)\theta}$$

Integrate both sides:
$$0 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big| f(\theta) - \sum_{-N}^{N} c_n e^{in\theta}\Big|^2 d\theta$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta - \frac{2}{2\pi}\mathrm{Re}\left\{\sum_{-N}^{N} \bar{c}_n \underbrace{\int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta}_{= 2\pi c_n}\right\}$$
$$+ \frac{1}{2\pi}\sum_{m,n=-N}^{N} c_m \bar{c}_n \underbrace{\int_{-\pi}^{\pi} e^{i(m-n)\theta} d\theta}_{= 2\pi \delta_{m,n}}$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta - \sum_{-N}^{N} |c_n|^2$$

Then letting $N \to \infty$ proves Bessel's Ineq.! //

Remark:
It turns out this is infact an equality, which is called Parseval's Equality.

Orthonormal sequence in a Hilbert space ⇒ Bessel
〃 basis 〃 〃 〃 ⇒ Parseval

Bessel's Ineq.: f is of finite energy ⇒ $\sum_{-\infty}^{\infty} |c_n|^2$: convergent

Corollary: The Fourier coef's $a_n, b_n, c_n, c_{-n} \to 0$ as $n \to \infty$.