Lecture 13: Fourier Series II

Note Title

☆ Convergence Thm's

A big question:

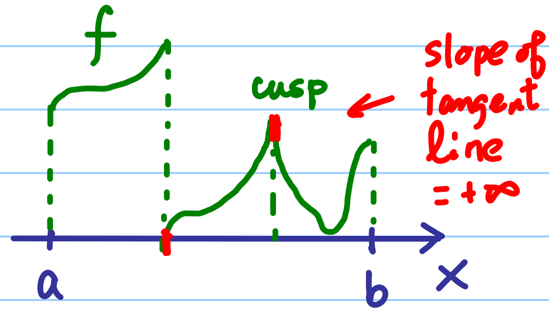
$$f(\theta) \overset{?}{=} \sum_{-\infty}^{\infty} c_n e^{in\theta} \quad \text{at every pt } \theta \in \mathbb{R}\,?$$

$$\text{or } f(\theta) \overset{?}{=} \lim_{N\to\infty} \sum_{-N}^{N} c_n e^{in\theta} \qquad "$$

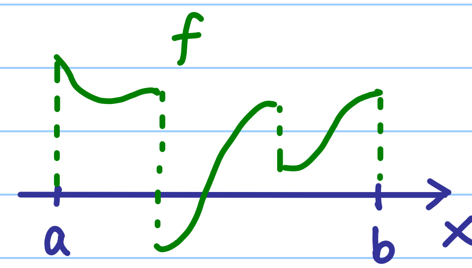
$\Rightarrow$ For a certain class of fcn's, this is Yes!

Def. $f \in PC[a,b]$ (piecewise continuous on $[a,b]$)
if (i) f is continuous on $[a,b]$ except perhaps at finitely many pts, say, $x_1, \cdots, x_k$; and
(ii) $f(x_j-) = \lim_{h\downarrow 0} f(x_j - h)$, $f(x_j+) = \lim_{h\downarrow 0} f(x_j+h)$ exist for each x_j, $j=1,\cdots,k$.

Def. $f \in PS[a,b]$ (piecewise smooth on $[a,b]$)
if (i) $f \in PC[a,b]$; and (ii) $f' \in PC(a,b)$.



$f \in PC[a,b]$
$\notin PS[a,b]$



$f \in PS[a,b]$

Remark: Classes of fcn's w.r.t. smoothness
(see, e.g., Davis & Rabinowitz: Methods of Numerical Integration, 2nd Ed., Dover, 2007)

Rougher $\longleftrightarrow$ smoother

$R[a,b]$, $BV[a,b]$, $C[a,b]$, $C^\alpha[a,b]$, $C^k[a,b]$, $A(I)$, $E(\mathbb{C})$, $\mathcal{P}_n$

Riemann Integrable; Bdd. Variation; $0<\alpha\le 1$ Lip α; $k\in\mathbb{N}$; $I \supset [a,b]$ Analytic; Entire; Polynom deg $\le n$

We'll work with c_n and complex F.S. $\sum_{-\infty}^{\infty} c_n e^{in\theta}$.

Recall $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\psi) e^{-in\psi}\, d\psi$

Consider the Nth partial sum of f

$$S_N[f](\theta) := \sum_{n=-N}^{N} c_n e^{in\theta}.$$

<u>Question</u>: $S_N[f](\theta) \xrightarrow[N\to\infty]{?} f(\theta)$, $\forall \theta \in \mathbb{R}$

$$S_N[f](\theta) = \sum_{-N}^{N} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\psi) e^{in(\theta-\psi)}\, d\psi$$

$n \mapsto -n$

$$= \frac{1}{2\pi}\sum_{-N}^{N}\int_{-\pi}^{\pi} f(\psi) e^{in(\psi-\theta)}\, d\psi$$

$\phi = \psi - \theta$

$$= \frac{1}{2\pi}\sum_{-N}^{N}\int_{-\pi-\theta}^{\pi-\theta} f(\theta+\phi) e^{in\phi}\, d\phi$$

f : 2π periodic

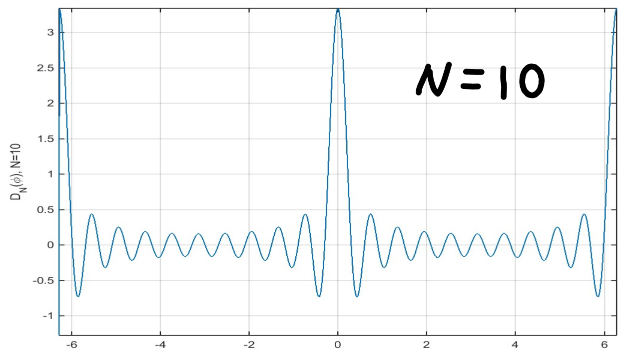
$$= \frac{1}{2\pi}\sum_{-N}^{N}\int_{-\pi}^{\pi} f(\theta+\phi) e^{in\phi}\, d\phi$$

$$= \int_{-\pi}^{\pi} f(\theta+\phi)\Big\{\underbrace{\frac{1}{2\pi}\sum_{-N}^{N} e^{in\phi}}_{=:\, D_N(\phi) \text{ the Nth Dirichlet kernel}}\Big\}\, d\phi$$

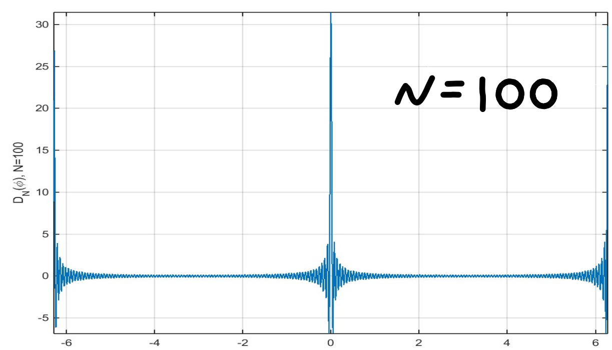
$$D_N(\phi) = \frac{1}{2\pi} e^{-iN\phi}\left(1 + e^{i\phi} + \cdots + e^{i2N\phi}\right)$$

$$= \frac{1}{2\pi} e^{-iN\phi}\, \frac{1-e^{i(2N+1)\phi}}{1-e^{i\phi}} = \frac{1}{2\pi}\, \frac{e^{i(N+1)\phi} - e^{-iN\phi}}{e^{i\phi} - 1}$$

$$= \frac{1}{2\pi}\, \frac{e^{i(N+\frac{1}{2})\phi} - e^{-i(N+\frac{1}{2})\phi}}{e^{i\phi/2} - e^{-i\phi/2}} = \frac{1}{2\pi}\, \frac{\sin(N+\frac{1}{2})\phi}{\sin\frac{\phi}{2}}$$



-2π 2π



-2π 2π

Lemma $\int_{-\pi}^{0} D_N(\theta)\,d\theta = \int_{0}^{\pi} D_N(\theta)\,d\theta = \frac{1}{2}$

i.e., $\int_{-\pi}^{\pi} D_N(\theta)\,d\theta = 1$.

(Proof) By the def. of $D_N(\theta)$, we can see

$$D_N(\theta) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{1}^{N}\cos n\theta \Rightarrow \int_{-\pi}^{0}\left(\frac{1}{2\pi} + \frac{1}{\pi}\sum_{1}^{N}\cos n\theta\right)d\theta$$

$$= \frac{\theta}{2\pi}\Big|_{-\pi}^{0} + \underbrace{\cancel{\sum_{1}^{N}\frac{\sin n\theta}{\pi n}\Big|_{-\pi}^{0}}}_{0}$$

$$= \frac{1}{2} \quad ///$$

Thm If f: 2π-periodic & $\in PS(\mathbb{R})$, then

$$\lim_{N\to\infty} S_N[f](\theta) = \frac{1}{2}\left[f(\theta-) + f(\theta+)\right], \ \forall\theta\in\mathbb{R}.$$

In particular, $\lim_{N\to\infty} S_N[f](\theta) = f(\theta)$ at every θ where f is continuous.

(Proof) We'll use the Lemma as

$$\frac{1}{2}f(\theta-) = f(\theta-)\int_{-\pi}^{0} D_N(\phi)\,d\phi, \quad \frac{1}{2}f(\theta+) = f(\theta+)\int_{0}^{\pi} D_N(\phi)\,d\phi$$

Now, $S_N[f](\theta) - \frac{1}{2}\left[f(\theta-) + f(\theta+)\right]$

$$= \int_{-\pi}^{\pi} f(\theta+\phi)D_N(\phi)\,d\phi - \int_{-\pi}^{0} f(\theta-)D_N(\phi)\,d\phi - \int_{0}^{\pi} f(\theta+)D_N(\phi)\,d\phi$$

$$= \int_{-\pi}^{0}\left[f(\theta+\phi) - f(\theta-)\right]D_N(\phi)\,d\phi + \int_{0}^{\pi}\left[f(\theta+\phi) - f(\theta+)\right]D_N(\phi)\,d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0}\frac{f(\theta+\phi) - f(\theta-)}{e^{i\phi} - 1}\left(e^{i(N+1)\phi} - e^{-iN\phi}\right)d\phi + \frac{1}{2\pi}\int_{0}^{\pi}\frac{f(\theta+\phi) - f(\theta+)}{e^{i\phi} - 1}\left(e^{i(N+1)\phi} - e^{-iN\phi}\right)d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\phi;\theta)\left(e^{i(N+1)\phi} - e^{-iN\phi}\right)d\phi \quad \text{———} \quad (*)$$

where $g(\phi;\theta) := \begin{cases} \dfrac{f(\theta+\phi)-f(\theta-)}{e^{i\phi}-1} & -\pi \le \phi \le 0 \\ \dfrac{f(\theta+\phi)-f(\theta+)}{e^{i\phi}-1} & 0 \le \phi \le \pi \end{cases}$.

g is a well-behaved smooth fcn on $[-\pi, \pi]$ except $\phi = 0$. But by l'Hôpital's rule,

$$\left.\begin{aligned} \lim_{\phi\downarrow 0} g(\phi;\theta) &= \lim_{\phi\downarrow 0} \frac{f'(\theta+\phi)}{ie^{i\phi}} = \frac{f'(\theta+)}{i} \\ \lim_{\phi\uparrow 0} g(\phi;\theta) &= \lim_{\phi\uparrow 0} \frac{f'(\theta+\phi)}{ie^{i\phi}} = \frac{f'(\theta-)}{i} \end{aligned}\right\} \text{both exist since } f \in PS(\mathbb{R}), \text{ i.e., } |f'(\theta\pm)| < \infty$$

$\Rightarrow g \in PC[-\pi, \pi]$.

So, by Corollary to Bessel's ineq.,

$$\hat{g}_n = c_n[g] = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\phi;\theta)e^{-in\phi}\,d\phi \xrightarrow[|n|\to\infty]{} 0.$$

Now $(*) = \hat{g}_{-(N+1)} - \hat{g}_N = c_{-(N+1)}[g] - c_N[g]$

$$\longrightarrow 0 \quad \text{as } N \to \infty \qquad ///$$

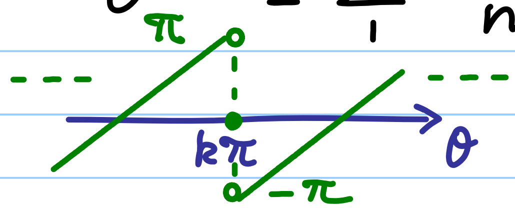
☆ Some applications

Example 1 $\quad f(\theta) = \theta \quad -\pi \le \theta \le \pi$, 2π periodic.

This f is clearly in $PS(\mathbb{R})$.

Recall its Fourier series. $\quad$ Conv. Thm.

$$\theta \sim 2\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n}\sin n\theta = \frac{1}{2}\left[(\theta+)+(\theta-)\right]$$



$$= \begin{cases} \theta & \text{if } -\pi < \theta < \pi \\ 0 & \text{if } \theta = \pm\pi \end{cases}$$

Example 2 $f(\theta) = |\theta|$, $-\pi \le \theta \le \pi$, 2π per.
This f is in $PS(\mathbb{R}) \cap C(\mathbb{R})$.
$\Rightarrow \frac{1}{2}[\,|\theta+| + |\theta-|\,] = |\theta| \quad \forall \theta \in \mathbb{R}$.

So, $$|\theta| = \frac{\pi}{2} - \frac{4}{\pi}\sum_1^\infty \frac{\cos(2n-1)\theta}{(2n-1)^2}, \quad \forall \theta \in [-\pi, \pi].$$

Example 3 $f(\theta) = \theta^2$, $-\pi \le \theta \le \pi$, 2π per.
again this is in $PS(\mathbb{R}) \cap C(\mathbb{R})$.
Computing its F.S. (an exercise!), we have
$$\theta^2 = \frac{\pi^2}{3} + 4\sum_1^\infty \frac{(-1)^n}{n^2}\cos n\theta, \quad \forall \theta \in [-\pi, \pi].$$
Set $\theta = \pi$.
$$\Rightarrow \pi^2 = \frac{\pi^2}{3} + 4\sum_1^\infty \frac{(-1)^n}{n^2}(-1)^n$$
$$\Leftrightarrow \sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$$
The Basel problem due to Euler!

You can also derive this using Example 2. (Exercise!)

Remark: $\exists$ 8 or 9 different proofs of the Basel problem; see the ref. page. Later in this course, I'll discuss another beautiful proof based on an eigenfcn expansion of Green's fcn for the Dirichlet-Laplacian on $[0, 1]$.