Lecture 19: Basics of L^2 Theory III

Note Title

☆ Other Types of L^2 spaces

(1) Weighted L^2 space

Let $w(x) \in (0, \infty)$ for a.e. $x \in [a,b]$.

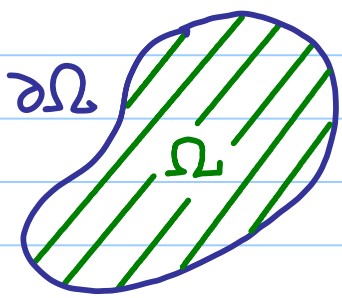
$$L^2_w[a,b] := \left\{ f : [a,b] \to \mathbb{C} \;\middle|\; \int_a^b |f(x)|^2 w(x)\, dx < \infty. \right\}$$

$$\langle f, g \rangle_w := \int_a^b f(x)\,\overline{g(x)}\, w(x)\, dx, \quad \|f\|_w := \sqrt{\langle f, f \rangle_w}$$

$\Rightarrow$ Used frequently in Sturm-Liouville Theory
Orthogonal Polynomials

(2) Higher-dimensions

Instead of the interval $[a,b]$, consider a domain $\Omega \subset \mathbb{R}^d$



$$L^2(\Omega) := \left\{ f : \Omega \to \mathbb{C} \;\middle|\; \int_\Omega |f(x)|^2 dx < \infty \right\}.$$

Thm $L^2(\Omega)$ is complete.

If $f \in L^2(\Omega)$, then $\exists \{f_n\} \subset L^2(\Omega)$ s.t. $\|f_n - f\| \to 0$ as $n \to \infty$. Moreover, one can take $\{f_n\} \subset C_0^\infty(\Omega) \subset C_0(\Omega)$.

have compact support in Ω.

☆ Hilbert Space

Def. a vector space $\mathcal{H}$ is called a Hilbert space if

(1) inner product is defined on elements in $\mathcal{H}$
(thus the norm is defined as $\|f\| = \sqrt{\langle f, f \rangle}$); and
(2) $\mathcal{H}$ is complete w.r.t. this norm.

Def. a vector space $\mathcal{B}$ is called a Banach space if

(1) a norm is defined on elements in $\mathcal{B}$; and
(2) $\mathcal{B}$ is complete w.r.t. this norm.

Def. a set $\mathcal{M}$ is called a metric space if a distance (i.e., metric) is defined among elem's of $\mathcal{M}$.

$$\left\{\begin{matrix}\text{Inner product}\\ \text{spaces}\end{matrix}\right\} \subset \left\{\begin{matrix}\text{Normed}\\ \text{spaces}\end{matrix}\right\} \subset \left\{\begin{matrix}\text{Metric}\\ \text{spaces}\end{matrix}\right\}$$

$$\cup \qquad\qquad \cup \qquad\qquad \cup$$

$$\left\{\begin{matrix}\text{Hilbert}\\ \text{spaces}\end{matrix}\right\} \subset \left\{\begin{matrix}\text{Banach}\\ \text{spaces}\end{matrix}\right\} \subset \left\{\begin{matrix}\text{Complete metric}\\ \text{spaces}\end{matrix}\right\}$$

e.g. $\searrow L^2(\Omega)$ $\qquad \searrow L^p(\Omega),\ 1 \le p \le \infty$ $\qquad \searrow \mathbb{R}$ with $d(x,y) = |x-y|/(1+|x-y|)$

See highly informative math.stackexchange.com posting!

Another important example of Hilbert space:

$$\ell^2(\mathbb{N}) := \left\{ \mathbb{c} = (c_j)_1^\infty,\ c_j \in \mathbb{C} \,\middle|\, \sum_1^\infty |c_j|^2 < \infty \right\}$$

For $\mathbb{c}, \mathbb{d} \in \ell^2(\mathbb{N})$, $\langle \mathbb{c}, \mathbb{d} \rangle := \sum_1^\infty c_j \bar{d}_j$, $\|\mathbb{c}\|_2 = \sqrt{\sum_1^\infty |c_j|^2}$.

Can show $\{\mathbb{c}_n\} \subset \ell^2(\mathbb{N})$: Cauchy $\Rightarrow \mathbb{c}_n \to {}^\exists \mathbb{c} \in \ell^2(\mathbb{N})$.

<u>Thm</u> Any Hilbert space is isomorphic to $\ell^2(\mathbb{N})$.
(Hence, $L^2(\Omega)$ is isomorphic to $\ell^2(\mathbb{N})$, which is referred to as the Riesz-Fischer Thm.)

<u>Def</u>. Two Hilbert spaces $\mathcal{H}$ & $\mathcal{H}'$ are said to be isomorphic to each other if $\exists$ a bijection $\Phi : \mathcal{H} \to \mathcal{H}'$ s.t.

(1) $\Phi(\alpha x + \beta y) = \alpha \Phi(x) + \beta \Phi(y)$, $\forall x, y \in \mathcal{H}$, $\forall \alpha, \beta \in \mathbb{C}$.

(2) $\langle x, y \rangle_{\mathcal{H}} = \langle \Phi(x), \Phi(y) \rangle_{\mathcal{H}'}$

(Proof) Suppose $\{\phi_n\}_1^\infty$ be an ONB for $\mathcal{H}$.
Then, we can consider a map $\Phi : \mathcal{H} \to \ell^2(\mathbb{N})$ by

$$\Phi f := \{\langle f, \phi_n \rangle\}_1^\infty, \quad \forall f \in \mathcal{H}.$$

Thanks to Parseval's equality $\|f\|_{\mathcal{H}}^2 = \sum_1^\infty |\langle f, \phi_n \rangle|^2$, so clearly $\Phi f \in \ell^2(\mathbb{N})$ and $\|f\|_{\mathcal{H}} = \|\Phi f\|_2$ (isometry)

(2) is also satisfied via Parseval, $\langle f, g \rangle_{\mathcal{H}} = \sum \langle f, \phi_n \rangle \overline{\langle g, \phi_n \rangle} = \langle \Phi f, \Phi g \rangle_2$

So, Φ is one-to-one from $\mathcal{H}$ to $\ell^2(\mathbb{N})$.
injection

Remains to show: Φ is also "onto" (surjection).
Take any $\mathbb{C} = (c_1, c_2, \cdots) \in \ell^2(\mathbb{N})$, i.e., $\sum |c_j|^2 < \infty$.
Then consider the following sequence in $\mathcal{H}$:
$f_1 = c_1\phi_1$, $f_2 = c_1\phi_1 + c_2\phi_2$, $\cdots$, $f_n = \sum_1^n c_j \phi_j$, $\cdots$
$\Rightarrow$ For $m > n$, $\|f_m - f_n\|_{\mathcal{H}}^2 \underset{\uparrow}{=} \sum_{j=n+1}^m |c_j|^2 \to 0$ as $m, n \to \infty$.
$\{\phi_j\}$: ONB

So, $\{f_n\}$: Cauchy in $\mathcal{H}$.
That implies that $f_n \to {}^{\exists} f \in \mathcal{H}$ thanks to its completeness.
For this f, we clearly have $\Phi f = \mathbb{C}$, and $\mathbb{C} \in \ell^2(\mathbb{N})$ was arbitrary. Hence we are done! ///

☆ The Dominated Convergence Thm

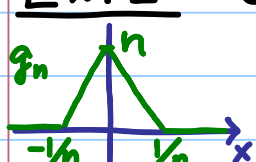
Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a domain, and let $\{g_n\}$, g, ϕ be fcns on Ω s.t.

$$\begin{cases} \text{(a) } \phi(\mathbf{x}) \geq 0 \text{ and } \int_\Omega \phi(\mathbf{x})\,d\mathbf{x} < \infty; \\ \text{(b) } |g_n(\mathbf{x})| \leq \phi(\mathbf{x}), \quad \forall n \in \mathbb{N}, \forall \mathbf{x} \in \Omega; \text{ and} \\ \text{(c) } g_n(\mathbf{x}) \xrightarrow[n\to\infty]{} g(\mathbf{x}), \quad \text{a.e. } \mathbf{x} \in \Omega. \end{cases}$$

Then, $\int_\Omega g_n(\mathbf{x})\,d\mathbf{x} \xrightarrow[n\to\infty]{} \int_\Omega g(\mathbf{x})\,d\mathbf{x}$.

<u>Ex. 1</u>: $d = 1$, $\Omega = \mathbb{R}$. $g_n(x) = e^{ix/n}\phi(x)$, $\phi \geq 0$, $\int_\Omega \phi < \infty$.
Then, $|g_n(x)| \leq |e^{ix/n}\phi(x)| = |\phi(x)| = \phi(x)$.
also, we see $g_n(x) \to \phi(x)$, so by the D.C. Thm,
$\lim \int_\Omega g_n(x)\,dx = \int_\Omega \lim g_n(x)\,dx = \int_\Omega \phi(x)\,dx < \infty$. //

<u>Ex. 2</u>: Consider the fcn in the figure.



$g_n(x) \to 0$ a.e., but $\int_\Omega g_n(x)\,dx = \frac{1}{2}\cdot n \cdot \frac{2}{n} = 1$.
So, $\lim \int g_n(x)\,dx = 1$. On the other hand,

there is no Φ satisfying (a), (b). Moreover, $\int_\Omega \lim g_n(x)\,dx = 0$. The D.C. Thm doesn't hold. The idea of δ fcn & the theory of distribution is necessary to deal with such an example!

Now, recall the following relationship :

$$\begin{tikzcd}
L^2\text{-Norm conv.} \arrow[rr, Rightarrow, shift left, "\textcircled{1}", "/" marking] \arrow[dr, Rightarrow, shift left, "/" marking, "\textcircled{3}"] & & \text{Pointwise conv.} \arrow[ll, Rightarrow, shift left, "/" marking, "\textcircled{2}"] \arrow[dl, Rightarrow, shift left, "/" marking] \\
& \text{Uniform conv.} \arrow[ul, Rightarrow, shift left, "\textcircled{4}"] \arrow[ur, Rightarrow, shift left] &
\end{tikzcd}$$

With a bit more assumption, however, we can show Pointwise conv. $\Rightarrow$ L^2-norm conv. as follows :

<u>Thm</u> Let $\{f_n\} \subset L^2(\Omega)$, $f_n \to f$ pointwise.
<u>If</u> $\exists \psi \in L^2(\Omega)$ s.t. $|f_n(x)| \le |\psi(x)|$ a.e. $x \in \Omega$,
<u>then</u> $f_n \to f$ in norm.

(Proof) $|f(x)| = \lim_{n\to\infty} |f_n(x)| \le |\psi(x)|$ a.e. $x \in \Omega$.
So, $|f_n(x) - f(x)|^2 \le (|f_n(x)| + |f(x)|)^2 \le |2\psi(x)|^2$

Now apply the D.C. Thm with $g_n(x) = |f_n(x) - f(x)|^2$, $g(x) = 0$, and $\Phi(x) = |2\psi(x)|^2$ to get :

$$\lim \underbrace{\int_\Omega g_n(x)\,dx}_{= \|f_n - f\|^2_{L^2(\Omega)}} = \int_\Omega \lim g_n(x)\,dx = \int_\Omega g(x)\,dx = 0$$

So, $\|f_n - f\|^2 \to 0$. ///

☆ Best Approximation in L^2

- If $\{\phi_n\}_1^\infty$ is an ONB of $L^2(\Omega)$, then $f = \sum_1^\infty \langle f, \phi_n \rangle \phi_n \quad \forall f \in L^2(\Omega)$.
- If $\{\phi_n\}$ is an ON set, but not complete in $L^2(\Omega)$, then $\forall f \in L^2(\Omega)$, $\exists$ residual error $f - \sum \langle f, \phi_n \rangle \phi_n$ and $\sum \langle f, \phi_n \rangle \phi_n \underset{\uparrow}{=} {}^{\exists}\tilde{f} \in L^2(\Omega)$.

Could be a finite subset of an ONB.

The last lemma in Lecture 18.

It turns out that $\tilde{f}$ is the best linear approx. of f in $L^2(\Omega)$ in the following sense:

Thm $\| f - \sum \langle f, \phi_n \rangle \phi_n \| \leq \| f - \sum \alpha_n \phi_n \|$ for arbitrary choice of $\{\alpha_n\}$ with $\sum |\alpha_n|^2 < \infty$.
$=$ holds iff $\alpha_n = \langle f, \phi_n \rangle, \ \forall n$.

(Proof) This is really the least squares approx.!

$$f - \sum \alpha_n \phi_n = \underbrace{f - \sum \langle f, \phi_n \rangle \phi_n}_{\perp \text{ to } \phi_j \in \{\phi_n\}} + \underbrace{\sum (\langle f, \phi_n \rangle - \alpha_n) \phi_n}_{\text{a linear comb. of } \{\phi_n\}}$$

$\because \langle f, \phi_j \rangle - \sum \langle f, \phi_n \rangle \underbrace{\langle \phi_n, \phi_j \rangle}_{= \delta_{nj}}$

So, the Pythagorian Thm applies:

$$\| f - \sum \alpha_n \phi_n \|^2 = \| f - \sum \langle f, \phi_n \rangle \phi_n \|^2 + \underbrace{\sum | \langle f, \phi_n \rangle - \alpha_n |^2}_{\geq 0}$$
$$\geq \| f - \sum \langle f, \phi_n \rangle \phi_n \|^2$$

and clearly $=$ holds iff $\alpha_n = \langle f, \phi_n \rangle$. ///

Cor. $\{\phi_n\}_1^\infty$: an ONB for $L^2(\Omega)$. Then for any $f \in L^2(\Omega)$, the Nth partial sum $\sum_1^N \langle f, \phi_n \rangle \phi_n$ is the best linear approx. in L^2-norm to f among all linear combinations of $\{\phi_1, \cdots, \phi_N\}$.

the least squares approx.

Note that $\{\phi_1, \cdots, \phi_N\}$ are selected independently from f.

If we choose N basis fcns dependent on f, i.e.,
$\{\phi_{\lambda_1}, \cdots, \phi_{\lambda_N}\} \subset \{\phi_n\}_1^\infty$ where $\lambda_1, \cdots, \lambda_N$ depend on f,
then $\sum_{j=1}^{N} \langle f, \phi_{\lambda_j} \rangle \phi_{\lambda_j}$ is better than $\sum_{j=1}^{N} \langle f, \phi_j \rangle \phi_j$ in general.

Example (Obvious) Say $N = 100$, $f(x) = \phi_{101}(x)$.
Then $f - \sum_{j=1}^{100} \langle f, \phi_j \rangle \phi_j = \phi_{101} \neq 0$.

But clearly $\lambda_1 = 101$, and $f = \phi_{101}$ is just a one term!
This way of approximation is called nonlinear approx.
In practice, $\{\lambda_1, \cdots, \lambda_N\}$ are chosen so that $|\langle f, \phi_{\lambda_j} \rangle|$ are the largest N expansion coeff's of f w.r.t. $\{\phi_n\}$.