Lecture 5: Natural Boundary Conditions

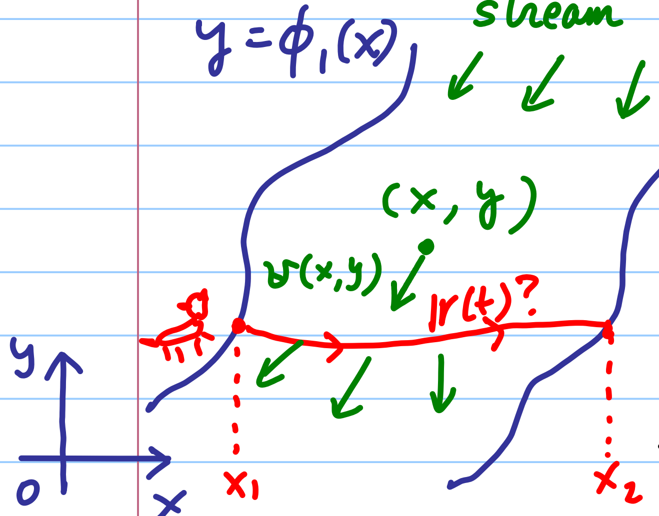
Note Title

☆ A Problem of Zermelo in Modified Form

Ernst Zermelo (1871–1953)

- Worked on C.V. for his Ph.D.
- Became more famous on the axiomatic set theory: "axiom of choice"
- Returned to the navigation problems in 1930's!

Given a stream, the left bank profile $y=\phi_1(x)$,
 " right " " $y=\phi_2(x)$



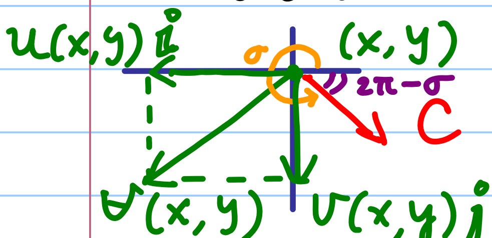
The velocity at (x,y) is specified as
$\mathbf{v}(x,y) = u(x,y)\,\mathbf{i} + v(x,y)\,\mathbf{j}$

So, we consider a vector field sandwiched between left & right banks.

A dog can swim with the const. speed c in still water. He (or she) wants to cross the stream with a min. amount of time.

<u>Question</u>: Where should he embark his swimming and what path should he take?

The actual velocity components of this dog are



$$(*)\begin{cases} u^* = u^*(x,y) = u(x,y) + c\cos\sigma \\ v^* = v^*(x,y) = v(x,y) + c\sin\sigma \end{cases}$$

Let $\mathbb{r}(t) = (x(t), y(t))^T$ be a path that dog takes.
Then, $u^* = \dot{x}$, $v^* = \dot{y}$
So, (*) $\Rightarrow (\dot{x} - u)^2 + (\dot{y} - v)^2 = c^2$

If $\mathbb{r}(t) = \mathbb{r}(\tau(t))$, then

$$\left(\frac{dx}{d\tau}\dot{\tau} - u\right)^2 + \left(\frac{dy}{d\tau}\dot{\tau} - v\right)^2 = c^2$$

Let's use $\tau(t) = x(t)$. Then $\dot{\tau} = \dot{x}$, $\frac{dx}{d\tau} = 1$.

$$\frac{dy}{d\tau} = \frac{dy}{dx} = y' \Rightarrow (\dot{x} - u)^2 + (y'\dot{x} - v)^2 = c^2$$

$$\Longleftrightarrow (1 + y'^2)\dot{x}^2 - 2(u + vy')\dot{x} + u^2 + v^2 - c^2 = 0$$

We are interested in $\frac{dt}{dx}$ (time behavior), and dividing the above by $\dot{x}^2 = \left(\frac{dx}{dt}\right)^2$ to get

$$(c^2 - u^2 - v^2)\left(\frac{dt}{dx}\right)^2 + 2(u + vy')\frac{dt}{dx} - (1 + y'^2) = 0$$

$$\Rightarrow \frac{dt}{dx} = \frac{-(u + vy') \pm \sqrt{(u + vy')^2 + (c^2 - u^2 - v^2)(1 + y'^2)}}{c^2 - u^2 - v^2}$$

$$= \frac{-(u + vy') \pm \sqrt{c^2(1 + y'^2) - (uy' - v)^2}}{c^2 - u^2 - v^2}$$

Integrating w.r.t. x from x_1 to x_2

$$t = \int_{x_1}^{x_2} \frac{-(u + vy') \pm \sqrt{c^2(1 + y'^2) - (uy' - v)^2}}{c^2 - u^2 - v^2}\, dx$$

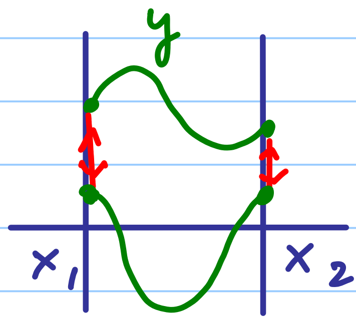
$\rightarrow$ min! (Find a curve $y = y^*(x)$ minimizing this t.) Note that x_1 and x_2 are not fixed!

With this generality, we cannot proceed too much from here. Note that a dog actually tries to do this kind of minimization. "dog curves"

☆ Natural B.C.'s for the 1D Problem

Consider $I = \int_{x_1}^{x_2} f(x, y, y')\,dx \to \min!$

But now $y = y(x)$ slides freely at $x = x_i$, $i = 1, 2$.



Yet, let's consider the variation of the form:

$y_\varepsilon = y(x, \varepsilon)$, $x \in [x_1, x_2]$.

$y^* = y_0 = y(x, 0)$ is the minimizer.

As before, set $\dfrac{\partial y_\varepsilon}{\partial \varepsilon} = \dfrac{\partial y(x,\varepsilon)}{\partial \varepsilon} =: \eta(x, \varepsilon)$.

But now, $\eta(x_i, \varepsilon) \neq 0$, $i = 1, 2$, unlike before.

What's the consequence?

$\Rightarrow$ The first term via Int. by Parts does not vanish and needs more thinking!

$$\delta I = \frac{dI}{d\varepsilon}(0) = \eta f_{y'} \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta \Big(\underbrace{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}}_{=0} \Big) dx$$

✗ 0 at x_1 & x_2

Yet, if we consider all the curves passing through fixed P_1 & P_2, the min. of $I(\varepsilon)$ should still lead to the E-L eqn.

$\Rightarrow \delta I = \eta f_{y'} \Big|_{x_1}^{x_2} = 0$ is a must.

$\Rightarrow f_{y'}(x_i) = 0$, $i = 1, 2$ since $\eta \in C^1[x_1, x_2]$ is arbitrary. This is called **natural bdry cond's.**

Example: Find a curve $y = y(x)$ on $0 \le x \le 1$

s.t. $I = \int_0^1 \underbrace{\sqrt{1+y'^2}}_{=f}\, dx \to \min!$

Ans. The natural bdy cond's yield

$$\frac{\partial f}{\partial y'}(x_i) = 0 \iff \frac{y'(x_i)}{\sqrt{1+y'(x_i)^2}} = 0$$

i.e., $y'(0) = y'(1) = 0$.

This is called the Neumann B.C. ← Carl Neumann (1832–1925)

Now, how about the E-L eqn.?

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\iff \frac{y''}{\sqrt{1+y'^2}} - \frac{y''\cdot(y')^2}{(1+y'^2)^{3/2}} = 0$$

So, we have $y'' = 0$ with $y'(0) = y'(1) = 0$.

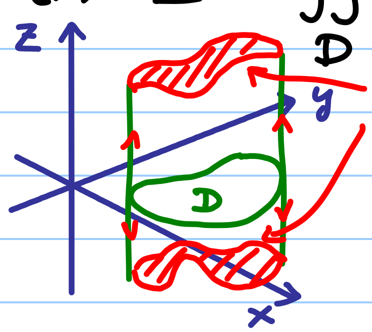
$\iff y = \text{const.}$ (a horizontal line).

and $\min. I = 1$. //

☆ Natural Bdy. Cond. for the 2D Problem

Consider the minimization of

$$(*) \quad I = \iint_D f(x, y, z(x,y), z_x, z_y)\, dx\, dy.$$



z(x,y) can slide along this cylinder wall!

D : a simply-connected domain $\subset \mathbb{R}^2$

$C = \partial D$: a rectifiable curve

As before, consider a variation $z_\varepsilon = z(x,y,\varepsilon)$ with $z^* = z_0 = z(x,y,0)$ being the minimizer of I. Also $\frac{\partial z_\varepsilon}{\partial \varepsilon} =: \zeta(x,y,\varepsilon)$

ζ does not necessarily vanish along C.

Recall the derivative of I using the Green's Thm:

$$\delta I = \frac{dI}{d\varepsilon}(0) = \iint_D \zeta\left(\frac{\partial f}{\partial z} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z_x} - \frac{\partial}{\partial y}\frac{\partial f}{\partial z_y}\right) dx\,dy$$

$$- \oint_C \zeta\left(\frac{\partial f}{\partial z_y} dx - \frac{\partial f}{\partial z_x} dy\right) = 0$$

$= 0$ on C may not hold.

However, the E-L eqn. $\frac{\partial f}{\partial z} - \frac{d}{dx}\frac{\partial f}{\partial z_x} - \frac{d}{dy}\frac{\partial f}{\partial z_y} = 0$ is a must since $z = z^*$ must yield a min. for each fixed bdy. So we must have

$$\oint_C \zeta\left(\frac{\partial f}{\partial z_y} dx - \frac{\partial f}{\partial z_x} dy\right) = 0 \quad \forall \zeta \in C^1(D)$$

$\Rightarrow$ F.L.C.V. $\quad \frac{\partial f}{\partial z_y} - \frac{\partial f}{\partial z_x}\frac{dy}{dx} = 0$ on C

$\Rightarrow$ 2D version of a nat. bdy. cond.

If $z = z^*$ minimizes I in $(*)$ without specifying bdy values on C, then it must satisfy both the E-L eqn. & the natural bdy. cond.

Example: Let $f = z_x^2 + z_y^2 = \|\nabla z\|^2$ and the min. of $I = \iint_D f\,dx\,dy$, i.e., the total gradient is minimized. Such a surface $z = z(x,y)$ must be smooth!

$$\frac{\partial f}{\partial z_x} = 2z_x, \quad \frac{\partial f}{\partial z_y} = 2z_y$$

So, the E-L. eqn $\dfrac{\partial f}{\partial z} - \dfrac{d}{dx}\dfrac{\partial f}{\partial z_x} - \dfrac{d}{dy}\dfrac{\partial f}{\partial z_y} = 0$

$\Leftrightarrow \quad z_{xx} + z_{yy} = 0$ in D, i.e.,

$\Delta z = 0$ in D $\Rightarrow$ Laplace's eqn!

Now how about the nat. bdy. cond.?

$$\frac{\partial f}{\partial z_y} - \frac{\partial f}{\partial z_x}\frac{dy}{dx} = 0 \quad \text{on } C$$

$\Leftrightarrow \quad 2z_y - 2z_x \cdot \dfrac{dy}{dx} = 0 \quad (**)$

Let's use a parametric form to describe C, i.e.,
$(x,y) = (x(s), y(s))$, $s \in J$: some 1D interval.

Then $\dfrac{dy}{dx} = \dfrac{dy/ds}{dx/ds} = \dfrac{\dot{y}}{\dot{x}}$, $\quad \dfrac{d}{ds}(\,) = (\dot{\,})$

Now $(**)$ becomes

$$z_y \dot{x} - z_x \dot{y} = 0 \quad (***)$$

Tangent vector to $C = \mathbb{t} := \dot{x}\,\mathbb{i} + \dot{y}\,\mathbb{j}$ $\quad \perp$

Normal " " " $= \boldsymbol{\nu} := -\dot{y}\,\mathbb{i} + \dot{x}\,\mathbb{j}$

So, $\dfrac{\partial z}{\partial \nu} := \boldsymbol{\nu} \cdot \nabla z = (-\dot{y}\,\mathbb{i} + \dot{x}\,\mathbb{j}) \cdot (z_x \mathbb{i} + z_y \mathbb{j})$

$= -\dot{y} z_x + \dot{x} z_y = 0$ by $(***)$!!

i.e., $\left.\dfrac{\partial z}{\partial \nu}\right|_C = 0$ the Neumann B.C.

On C, $z = z^*$ must be flat in the normal directions.