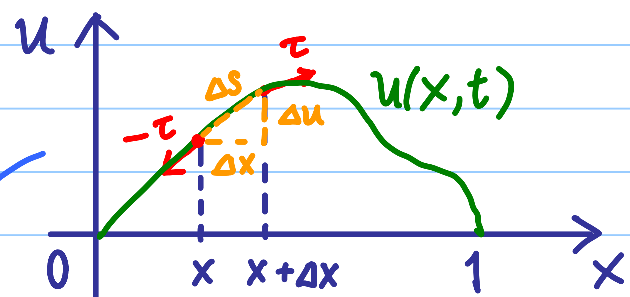
Lecture 7: Basics of PDEs I

Note Title

☆ The Vibrating String: Vectorial Approach

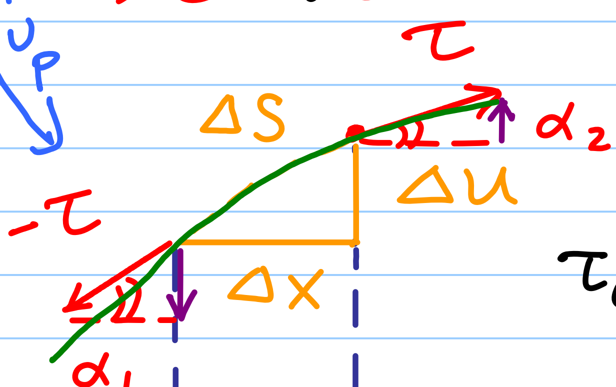
We already derived a wave eqn. via Calc. of Var. Here, we shall try a more traditional vectorial approach to see that we reach the same wave eqn.



For the future convenience, let's consider a flexible string between two fixed pts $x=0$ & $x=1$.

Assume that the string is perfectly flexible, i.e., only tensile forces can be transmitted along the tangential direction.

$\Rightarrow$ The tension acts with the same magnitude at any part of the string along the tan. dir.



The total external forces exerted upon the element Δs (u-direction) is

$$\tau_{(u)} = \tau(\sin\alpha_2 - \sin\alpha_1)$$

Note that $\sin\alpha_1 = \dfrac{du(x)}{ds} = \dfrac{du}{dx}\dfrac{dx}{ds} = \dfrac{du}{dx}\dfrac{1}{\sqrt{1+\left(\frac{du}{dx}\right)^2}}$

So far, our argument doesn't include "t". So for the moment, we write $\frac{du}{dx}$ instead of $\frac{\partial u}{\partial x}$. as before, $\sqrt{1+\left(\frac{du}{dx}\right)^2} = 1 + \underbrace{\frac{1}{2}\left(\frac{du}{dx}\right)^2 + O\left(\left(\frac{du}{dx}\right)^4\right)}_{\text{negligible}}$

Then $\sin\alpha_1 = \frac{du}{dx}\frac{1}{\sqrt{1+(\frac{du}{dx})^2}} \approx \frac{du}{dx} = u'(x)$

How about $\sin\alpha_2$?

$$\sin\alpha_2 = \frac{du}{ds}(x+\Delta x) \underset{\uparrow\, ds\approx dx}{\approx} \frac{du}{dx}(x+\Delta x) \overset{\text{Mean Value Thm.}}{=} u'(x)+u''(\xi)\Delta x$$

$\exists\, \xi \in [x, x+\Delta x]$

So, $\tau_{(u)} = \tau(\sin\alpha_2 - \sin\alpha_1) \approx \tau\, u''(\xi)\Delta x$

$= \tau\, u_{xx}(\xi)\Delta x$ — ①

Now, the inertial force exerted by the same element Δs is $\rho\,\Delta s\, u_{tt}$ = mass × acceleration — ②

By equating ① & ② and dividing by Δx, we have

$$\rho\frac{\Delta s}{\Delta x} u_{tt} = \tau\, u_{xx}(\xi)$$

Taking the limit $\Delta x \to 0$, we have $\xi \to x$, $\frac{\Delta s}{\Delta x} \to \frac{ds}{dx} \approx 1$

assuming $\frac{\partial u}{\partial x}$: small.

$\Rightarrow$ (*) $\rho\, u_{tt} = \tau\, u_{xx}$ or

$$u_{tt} = \frac{\tau}{\rho}u_{xx} = c^2 u_{xx}, \quad c = \sqrt{\frac{\tau}{\rho}}$$

1D wave eqn. again! velocity

<u>If</u> $\exists$ external force of density $f(x,t)$ acts on the unit length of the string, <u>then</u>

$$\tau_{(u)} \approx \tau\, u_{xx}(\xi)\Delta s + f(x,t)\Delta s.$$

So, (*) becomes

$$(**) \quad u_{tt} = \frac{\tau}{\rho}u_{xx} + \frac{1}{\rho}f(x,t)$$

Similarly, <u>if</u> we want to use Hamilton's principle <u>then</u> the action becomes

$$\frac{1}{2}\int_{t_1}^{t_2}\int_0^1 [\rho u_t^2 - \tau u_x^2 + 2fu]\,dx\,dt \to \min!$$

i.e., $V = \frac{1}{2}\int_0^1 \tau u_x^2\,dx - \underbrace{\int_0^1 f(x,t)\,u(x,t)\,dx}_{=\text{ work}}$

$\Rightarrow$ The E-L eqn. leads to $(**)$ again!

☆ Toward a solution of the string (1D Wave) eqn.
— Separation of variables + Superposition Principle

$$u_{tt} = c^2 u_{xx}, \quad c^2 = \frac{\tau}{\rho}$$

Do the coordinate transf. via $\sqrt{\tau}\, t \to$ new t
$\sqrt{\rho}\, x \to$ new x

to get $\quad u_{tt} = u_{xx} \quad (*)$

Let's consider this string eqn. on $[0,\pi]$ with

$$\begin{cases} \text{B.C. (Dirichlet)} & u(0,t) = u(\pi,t) = 0, \quad \forall t \\ \text{I.C.} & \begin{cases} u(x,0) = \phi(x) \in C_0^1[0,\pi], \text{ i.e., } \phi \in C^1[0,\pi] \ \& \ \phi(0)=\phi(\pi)=0. \\ u_t(x,0) = 0 \end{cases} \end{cases}$$

Let's reduce this PDE to a set of ODEs since the latter are easier to solve!

Assume the solution can be written in the form

$$u(x,t) = X(x)\cdot T(t) \quad (**)$$

Separation of variables
aka Daniel Bernoulli's separation method

Now, the B.C. above leads to $X(0) = X(\pi) = 0$
whereas the 0 velocity I.C. " " $T'(0) = 0$

Let's plug $(**)$ into $(*)$!

$$(X\cdot T)_{tt} = X\cdot T'' = (X\cdot T)_{xx} = X''\cdot T$$

For a region of (x,t) where $u(x,t) \neq 0$,
divide the above by $u = X \cdot T$ to get

$$\underbrace{\frac{T''(t)}{T(t)}}_{\text{a fcn of } t} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{a fcn of } x} = \text{must be a const.} =: \lambda$$

$$\Longleftrightarrow \begin{cases} X'' - \lambda X = 0 & \text{—— ①} \quad \text{B.C. } X(0) = X(\pi) = 0 \\ T'' - \lambda T = 0 & \text{—— ②} \quad \text{I.C.: } T'(0) = 0. \end{cases}$$

①: The general sol. $X(x) = a_1 e^{\sqrt{\lambda}x} + a_2 e^{-\sqrt{\lambda}x}$
The characteristic eqn $\quad a_1, a_2 \in \mathbb{R}$

$$r^2 - \lambda = 0 \implies r = \pm\sqrt{\lambda}.$$

Case I: $\lambda > 0$. Then $\pm\sqrt{\lambda} \in \mathbb{R}$

So, $X(0) = a_1 + a_2 = 0 \iff a_2 = -a_1$

$$X(\pi) = a_1 e^{\sqrt{\lambda}\pi} + a_2 e^{-\sqrt{\lambda}\pi}$$
$$= a_1 (e^{\sqrt{\lambda}\pi} - e^{-\sqrt{\lambda}\pi}) = 0$$
$$\iff a_1 = 0 = a_2 \ \#$$

Case II: $\lambda = 0$. Then $X'' = 0$, so $X(x) = a_1 + a_2 x$.
$X(0) = a_1 = 0$, $X(\pi) = a_2 \pi = 0 \iff a_2 = 0$ #

Case III: $\lambda < 0$. Let's set $\lambda = -\mu^2$, $\mu \in \mathbb{R}$
Then we get $X(x) = a_1 \cos\mu x + b_1 \sin\mu x$
$\exists a_1, b_1 \in \mathbb{R}$.

$X(0) = a_1 = 0$, $X(\pi) = b_1 \sin\mu\pi = 0$.
Since $b_1 \neq 0$, we have $\sin\mu\pi = 0$.
i.e., $\mu \in \mathbb{Z} \setminus \{0\}$. But the negative sign can be absorbed in the arbitrarity of b_1,
So, it suffices to consider $\mu = k \in \mathbb{N}$.
i.e., $\mu = 1, 2, \cdots$

$\Rightarrow X(x) = b_1 \sin kx$ —— ①′ $b_1 \in \mathbb{R}$

②: $T'' - \lambda T = 0 \Rightarrow T'' + k^2 T = 0$, $k \in \mathbb{N}$.

So, $T(t) = a_2 \cos kt + b_2 \sin kt$

$T'(t) = -a_2 k \sin kt + b_2 k \cos kt$

$T'(0) = b_2 k = 0 \iff b_2 = 0$.

$\Rightarrow T(t) = a_2 \cos kt$ —— ②′ $a_2 \in \mathbb{R}$

From ①′ & ②′, we see $u_k(x,t) = b_k \sin kx \cos kt$ $b_k \in \mathbb{R}$ for each $k \in \mathbb{N}$ is a sol. to the string eqn. with those B.C. and I.C.′ (velocity I.C.)

The remaining problem: how to satisfy the other half of I.C., i.e., $u(x,0) = \phi(x)$?

But, the only constraint is $\phi \in C_0^1[0,\pi]$.

We cannot assume $\phi(x) = C \cdot \sin kx$, $\exists C \in \mathbb{R}$

The idea: Since $u_{tt} = u_{xx}$ is linear,

if $\phi(x) \approx \sum_{k=1}^{n} b_k \sin kx$ (i.e., a trig. poly.)

then $u(x,t) \approx \sum_{k=1}^{n} b_k \sin kx \cos kt$.

Since $k = 1, 2, \cdots, n, n+1, \cdots$, why not consider $\sum_{k=1}^{\infty} b_k \sin kx \rightarrow$ Fourier (sine) series

Many questions naturally arise:

- For a given $\phi(x) \in C_0^1[0,\pi]$, how to compute $\{b_k\}$?
- How fast does b_k decay? (faster decay → better approx.)
- What is the mode of convergence? uniform? pointwise? norm?
- Can we weaken the condition on $\phi \in C_0^1[0,\pi]$? say, $\phi \in C_0[0,\pi]$?
- Can we get faster decay if $\phi \in C_0^m[0,\pi]$, $m > 1$?

These will be addressed later when we discuss Fourier Series!

<u>Remark</u>: $\exists$ linear PDEs where the sep. of var's does not work. See, e.g., an article by V.G. Papanicolaou in the ref. page.

<u>Remark on B.C./I.C. in general</u>

- For ODEs, e.g., $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$ of order n, a general sol. contains n arbitrary const's. e.g., $y = y(x, c_1, \cdots, c_n)$
  $\Rightarrow$ To resolve $c_1, \cdots, c_n$, we need n constraints often in the form of B.C.'s and/or I.C.'s, e.g., $y(x_0) = y_0,\ y'(x_0) = y_1, \cdots, y^{(n-1)}(x_0) = y_{n-1}$

- In the case of PDEs, instead of const's, we get arbitrary fcn's!
  e.g., $u_{tt} = c^2 u_{xx}$
  $\Rightarrow$ a general sol. is

$$u(x,t) = f(x-ct) + g(x+ct)$$

where f, g are arbitrary C^2 fcns.

$\Rightarrow$ To get a sol. describing a specific process/phenomenon, we need rather severe B.C.'s and/or I.C.'s, i.e., those involving fcns!