Lecture 8 : Basics of PDEs II

Note Title

☆ The Uniqueness of the String Eqn.

Once a sol. is obtained via, e.g., sep. of var's, can we guarantee its uniqueness?
If yes, then we have peace of mind!

$$(*)\begin{cases} u_{tt} = \frac{\tau}{\rho} u_{xx} + \frac{1}{\rho} f(x,t), \quad x \in [x_1, x_2], \ t \geq t_0. \\ \underline{\text{B.C.}} \quad u(x_1,t) = u(x_2,t) = 0 \quad (\text{Dirichlet}) \\ \underline{\text{I.C.}} \quad \begin{cases} u(x,t_0) = \phi(x), & \phi(x_i) = 0, \ i=1,2 \\ u_t(x,t_0) = \psi(x), & \psi(x_i) = 0, \quad " \end{cases} \end{cases}$$

<u>Thm</u> Let $D = [x_1, x_2] \times [t_0, \infty) \subset \mathbb{R}^2$
Let $u \in C^2(D)$. <u>If</u> $u(x,t)$ satisfies $(*)$, <u>then</u> it is the only fcn in D with these properties!

<u>(Proof)</u> Let's assume $\exists v \in C^2(D)$ s.t. $u \not\equiv v$.
Define $\zeta(x,t) := u(x,t) - v(x,t)$.
Then ζ is a sol. of

$$(**)\begin{cases} \zeta_{tt} = \frac{\tau}{\rho} \zeta_{xx} \quad \text{in } D \\ \zeta(x_i, t) = 0, \quad i=1,2 \\ \zeta(x, t_0) = 0 = \zeta_t(x,t_0) \end{cases}$$

easy to derive these since $(*)$ is linear!

To SHOW: ζ satisfying $(**) \equiv 0$ in D.
This is physically obvious since nothing really happens for ζ with 0 B.C. & I.C.
Yet, we need to prove this mathematically.

Consider the total energy (= kinetic + potential) at any time $t, \geq t_0$.

Then, "nothing happens" means that

$$(***)\ \int_{x_1}^{x_2} \underbrace{[\rho \zeta_t^2(x,t_1) + \tau \zeta_x^2(x,t_1)]}_{\text{Total energy at }(x,t_1).} dx = 0$$

If we can show this, then we can say
$\zeta(x,t) \equiv 0$ in D.

To show $(***)$, let's consider

$$(\star)\ \int_{t_0}^{t_1}\int_{x_1}^{x_2} \underbrace{\zeta_t [\rho \underbrace{\zeta_{tt} - \tau \zeta_{xx}}_{=0 \text{ from } (**),\text{ so}}]}_{} \, dx\, dt = 0$$

$$= \frac{1}{2}\frac{\partial}{\partial t}[\rho \zeta_t^2 + \tau \zeta_x^2] - \frac{\partial}{\partial x}(\tau \zeta_t \cdot \zeta_x)$$

$$\left(\because \frac{1}{2}(2\rho \zeta_t \zeta_{tt} + 2\tau \zeta_x \zeta_{xt}) - \tau \zeta_{tx}\zeta_x - \tau \zeta_t \zeta_{xx} = \zeta_t[\rho \zeta_{tt} - \tau \zeta_{xx}] \right)$$

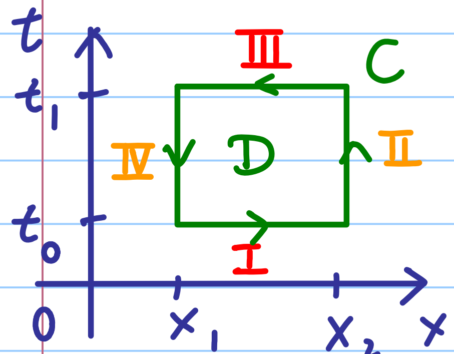
So, $(\star) = \int_{t_0}^{t_1}\int_{x_1}^{x_2} [\frac{1}{2}\frac{\partial}{\partial t}(\rho \zeta_t^2 + \tau \zeta_x^2) - \frac{\partial}{\partial x}(\tau \zeta_t \zeta_x)]\, dx\, dt$

Now, $\int_{t_0}^{t_1}\int_{x_1}^{x_2} [\ ]\, dx\, dt = \iint_D [\ ]\, dx\, dt$, so let's use

Green's Thm: $\iint_D \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial t}\right) dx\, dt = \oint_C (f_1 dt - f_2 dx)$

Set $f_1 = -\tau \zeta_t \zeta_x$, $f_2 = \frac{1}{2}(\rho \zeta_t^2 + \tau \zeta_x^2)$.

Then, $(\star) = \oint_C \left(-\tau \zeta_t \zeta_x\, dt - \frac{1}{2}(\rho \zeta_t^2 + \tau \zeta_x^2)\, dx\right)$



$$= \int_{\text{II}+\text{IV}} (-\tau \zeta_t \zeta_x)\, dt + \int_{\text{I}+\text{III}} -\frac{1}{2}(\rho \zeta_t^2 + \tau \zeta_x^2)\, dx$$

on I ($t = t_0$, $x_1 \le x \le x_2$), $\zeta = 0 = \zeta_t$ (I.C.)

II ($t_0 \le t \le t_1$, $x = x_2$)
IV ($t_0 \le t \le t_1$, $x = x_1$) $\Big\}$ $\zeta = 0$ (B.C.) so, $\zeta_t = 0$.

Hence $(\star) = -\frac{1}{2}\int_{\text{III}} (\rho \zeta_t^2 + \tau \zeta_x^2)\, dx$

$$= \frac{1}{2}\int_{x_1}^{x_2} (\rho \zeta_t^2(x,t_1) + \tau \zeta_x^2(x,t_1))\, dx$$

$$= 0, \quad \forall t_1 \geq t_0$$

$\Leftrightarrow \zeta_t(x,t_1) = 0 = \zeta_x(x,t_1), \quad \forall (x,t_1) \in D$

$\Leftrightarrow \zeta(x,t) \equiv$ a const. in D

By the B.C., $\zeta(x_i,t) = 0$, $i = 1, 2$.

this const. $= 0$, i.e., $\zeta(x,t) \equiv 0$ in D. ///

☆ Heat Conduction without Convection

We only consider heat conduction in a body whose material is homogeneous and isotropic.

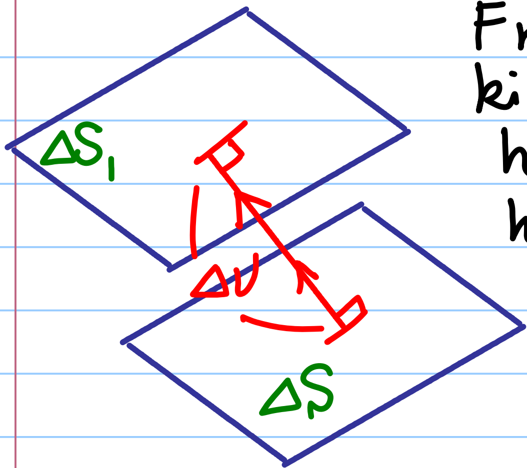
(no inhomogenuity e.g., no void, no composite material) (no preferred direction)

Remark: However, anisotropic diffusion has become an important & interesting topic in image processing (e.g., "Perona-Malik" model) //

Let the quantity of heat ΔQ penetrating an arbi. chosen surface ΔS within the medium in unit time, i.e., ΔQ = flux of heat through ΔS.

Consider another surface element $\Delta S_1 \,//\, \Delta S$ with the distance $\Delta \nu$. Assume the temperature

$$\begin{cases} u = u(x,y,z,t) = \text{Const. on } \Delta S, \\ u_1 = u(x+\Delta x, y+\Delta y, z+\Delta z, t+\Delta t) = \text{Const}' \text{ on } \Delta S_1 \end{cases}$$



From experimental thermodynamics + kinetic theory of molecular motion, heat will flow from the surface of high temp. to that of low temp. in the direction $\perp$ to these surfaces.
(The First Law of Thermodynamics)

Let $\Delta u = u_1 - u$. Then we have

$$\Delta Q \cdot \Delta\nu \propto \Delta u \cdot \Delta S$$
$$= k \, \Delta u \, \Delta S \quad \text{——} \quad (*)$$

thermal conductivity

k = flux of heat when $\Delta u = 1$, $\Delta S = 1$, & $\Delta\nu = 1$.
Note that k in general depends on material and temp. We won't deal with $k = k(u)$.
Instead, we will only deal with $k = $ Const.

From $(*)$, we have $\Delta Q = k \dfrac{\Delta u}{\Delta \nu} \Delta S$

$$\to k \frac{\partial u}{\partial \nu} \Delta S \quad \text{as } \Delta\nu \to 0.$$

normal deriv. of u on ΔS

So, $Q = k \displaystyle\iint_S \frac{\partial u}{\partial \nu} \, dS$

$$= k \iint_S \boldsymbol{\nu} \cdot \nabla u \, dS$$

$\hookrightarrow$ the unit normal vector on S

Via the divergence thm for a vector field in $\mathbb{R}^3$

$$\iint_S \mathbb{F} \cdot \boldsymbol{\nu} \, dS = \iiint_V \nabla \cdot \mathbb{F} \, dV$$

and setting $\mathbb{F} = \nabla u$, $V = \Delta v$, we have

$$Q = k \iint_S \nabla u \cdot \nu \, dS = k \iiint_{\Delta v} \nabla \cdot \nabla u \, dV$$

$$= k \iiint_{\Delta v} \varDelta u \, dV$$

$\varDelta = \nabla \cdot \nabla$: Laplacian

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$= \nabla^2$$

Remark: In LaTeX, I would recommend

\Delta for the differential inclement e.g., Δt

\varDelta for the Laplacian, e.g., $\varDelta u$ //

Apply the Mean Value Thm now to get

$$Q = k \nabla^2 u(\xi_i, \eta_j, \zeta_k, t) \Delta v \qquad \text{—— ①}$$

for $\exists (\xi_i, \eta_j, \zeta_k) \in \Delta v$

1D version:

$$f(c) = \frac{F(b) - F(a)}{b - a}, \ {}^{\exists} c \in I = (a, b), \ F' = f$$

$$\Longleftrightarrow (b-a) f(c) = F(b) - F(a), \ {}^{\exists} c \in I$$

$$\Longleftrightarrow \mathrm{vol}(I) f(c) = \int_I f(x) \, dx, \ {}^{\exists} c \in I$$

Now you can see how to get ①

On the other hand, u in Δv is either increased or decreased by the accumulation or loss of quantity of heat in Δv at the rate $\langle u_t \rangle$ (= the spatial average of u_t over Δv). So, the flux of heat into/from Δv is:

$$\rho \sigma \langle u_t \rangle \Delta v \qquad \text{—— ②}$$

σ = specific heat of the medium
= amout of heat necessary to raise the temp. of a unit mass of medium by one unit temp. in unit time.

Note in ②, $\rho \Delta v$ = mass of the medium.

Equating ① = ② and letting $\Delta v \downarrow 0$, we have

$$\rho \sigma u_t = k \nabla^2 u$$

or

$$u_t = \frac{k}{\rho\sigma} \nabla^2 u = \alpha^2 \Delta u, \quad \alpha^2 = \frac{k}{\rho\sigma}$$

Heat (or Diffusion) Eqn.!

If $\exists$ heat source or sink at (x,y,z) at time t, say, $f(x,y,z,t)$, then the above PDE becomes

$$u_t = \alpha^2 \Delta u + \frac{1}{\rho\sigma} f(x,y,z,t)$$

For the stationary temp. distrib., i.e., $u_t \equiv 0$, we have

$$\Delta u = -\frac{1}{k} f \quad \text{(Poisson's eqn.)}$$

If $\exists$ no f, then this reduces to

$$\Delta u = 0 \quad \text{(Laplace's eqn.)}$$