Lecture 14: Fourier Series III

Note Title

☆ Derivatives, Integrals, & Uniform Convergence

Let $f(\theta) \sim \sum c_n e^{in\theta} = \sum \hat{f}_n e^{in\theta}$.

And let's write

$$f'(\theta) \sim \sum c_n' e^{in\theta} = \sum \hat{f}_n' e^{in\theta}$$

$$F(\theta) = \int_0^\theta f(\phi)\,d\phi \sim \sum C_n e^{in\theta} = \sum \hat{F}_n e^{in\theta}$$

What are the relationship between c_n', C_n and c_n?

<u>Answer</u>: $c_n' = in\,c_n$, $C_n = \dfrac{c_n}{in}$ $(n \neq 0)$

You can see $\dfrac{d}{d\theta}$: roughening

$\int^\theta \cdot\, d\phi$: smoothing

More precisely,

<u>Thm</u> Let f: 2π-periodic & $\in C(\mathbb{R}) \cap PS(\mathbb{R})$.

Then, $c_n' = in\,c_n$ (or $\hat{f}_n' = in\hat{f}_n$).

In terms of a_n, b_n, we have

$a_n' = n b_n$, $b_n' = -n a_n$ (Exercise!)

(Proof)

$$c_n' = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(\theta) e^{-in\theta}\,d\theta$$

$$\overset{\text{Int. by Parts}}{=} \frac{1}{2\pi}\left\{ f(\theta)e^{-in\theta}\Big|_{-\pi}^{\pi} + in\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta \right\}$$

$$= \frac{1}{2\pi}\{\underbrace{f(\pi)e^{-in\pi}}_{(-1)^n f(\pi)} - \underbrace{f(-\pi)e^{in\pi}}_{(-1)^n f(-\pi) = (-1)^n f(\pi)}\} + in\,c_n$$

since f: 2π-per. & in $C(\mathbb{R})$.

$$= in\,c_n \quad ///$$

As for the F.S. of an antiderivative of f, note first that

an antiderivative of a periodic fcn $\neq$ a periodic fcn in general!

e.g., $f(\theta) \equiv 1$ is a periodic fcn, but its antiderivative $F(\theta) = \theta$ is not periodic.

The key is $c_0 = \hat{f}_0 = 0$ or not.

Since $\int_0^\theta c_n e^{in\phi} d\phi$ is 2π-periodic if $n \neq 0$, we can see $F(\theta)$ is 2π-periodic if $c_0 = \hat{f}_0 = 0$.

<u>Thm</u> Let $f \in PC(\mathbb{R})$, 2π-periodic, and $F(\theta) = \int_0^\theta f(\phi)\, d\phi$. If $c_0 = \hat{f}_0 = 0$, then

$$F(\theta) = C_0 + \sum_{n \neq 0} \frac{c_n}{in} e^{in\theta} = \frac{A_0}{2} + \sum_1^\infty \left(\frac{a_n}{n} \sin n\theta - \frac{b_n}{n} \cos n\theta \right)$$

where $C_0 = \hat{F}_0 = \frac{A_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\theta)\, d\theta$.

(Proof) $F(\theta + 2\pi) - F(\theta) = \int_\theta^{\theta+2\pi} f(\phi)\, d\phi \underset{f:\ 2\pi\text{-periodic}}{=} \int_{-\pi}^{\pi} f(\phi)\, d\phi$
$= 2\pi c_0 = 0$.

So, $F(\theta)$ is also 2π-periodic!

Moreover, $F \in C(\mathbb{R}) \cap PS(\mathbb{R})$ because $f \in PC(\mathbb{R})$.

Hence, the F.S. of F at $\theta = F(\theta)$ $\forall \theta \in \mathbb{R}$.

By applying the previous Thm to $F(\theta)$, we get

$$c_n = \hat{f}_n = in\, C_n = in \hat{F}_n \Rightarrow \hat{F}_n = \frac{\hat{f}_n}{in}, \quad n \neq 0$$

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These two thms suggest:

$\frac{d}{d\theta}$: high freq. coef ↑ ; $\int^\theta \cdot\, d\phi$: high freq. coef. ↓

The previous conv. thm, i.e., $S_N[f](\theta) \to \frac{1}{2}[f(\theta-)+f(\theta+)]$ $\forall \theta \in \mathbb{R}$ for $f \in PS(\mathbb{R})$, 2π-periodic, was about the pointwise convergence of F.S.
Now we want to have a thm of absolute & uniform convergence!

__Def.__ Suppose $\sum_1^\infty g_n(x)$ converges to $g(x)$ on $x \in S$, S: some set. Then if $\sum_1^\infty |g_n(x)|$ converges for $x \in S$ too, $\sum_1^\infty g_n(x)$ is said to converge absolutely on S. __If__ $\sup_{x \in S} \left| g(x) - \sum_1^N g_n(x) \right| \xrightarrow[N\to\infty]{} 0$ $\forall x \in S$, __then__ $\sum_1^\infty g_n(x)$ is said to converge uniformly to $g(x)$ on S.

__Ex.__ $g_n(x) = \dfrac{2(-1)^{n+1}}{n} \sin nx$, $S = [-\pi, \pi]$.
$\Rightarrow \sum g_n(x)$, i.e., the F.S. of $g(x) = x$ (2π-per.), does not converge uniformly to $g(x)$ because the uniform limit of a continuous fcn must be continuous, but g is discontinuous at $x = \pm\pi$.

To check the abs. & unif. conv. of a series, we can use the Weierstrass M-test (aka the Comparison Test): __If__ $\exists M_n \geq 0$ is a seq. s.t. $|g_n(x)| \leq M_n$, $\forall x \in S$ & $\sum_1^\infty M_n < \infty$, __then__ $\sum_1^\infty g_n(x)$ conv. abs. & unif. on S.

So, in our case of F.S., because $|c_n e^{in\theta}| = |c_n|$, if $\sum_{-\infty}^{\infty} |c_n| < \infty$, then the F.S. conv. abs. & unif., that is,

<u>Thm</u> (Sufficiency for abs. & unif. conv.)

<u>If</u> $f \in C(\mathbb{R}) \cap PS(\mathbb{R})$, 2π-periodic, <u>then</u> the F.S. of f converges to f abs. & unif. on $\mathbb{R}$.

(Proof) To show: $\sum_{-\infty}^{\infty} |\hat{f}_n| < \infty$.

(Note that we already know $\sum |\hat{f}_n|^2 < \infty$ thanks to Bessel's inequality!)

Because of the cond. on f, we have

$$\hat{f'}_n = in\hat{f}_n \Rightarrow |\hat{f}_n| = \left|\frac{\hat{f'}_n}{n}\right| \text{ for } n \neq 0.$$

Bessel's inequality applied to f' gives us

$$\sum_{-\infty}^{\infty} |\hat{f'}_n|^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(\theta)|^2 d\theta < \infty$$

So, $$\sum_{-\infty}^{\infty} |\hat{f}_n| = |\hat{f}_0| + \sum_{n\neq 0} \left|\frac{\hat{f'}_n}{n}\right| \le |\hat{f}_0| + \Big(\sum_{n\neq 0} \frac{1}{n^2}\Big)^{1/2} \cdot \Big(\sum_{n\neq 0} |\hat{f'}_n|^2\Big)^{1/2}$$

↑ Cauchy–Schwarz!

$$= |\hat{f}_0| + \frac{\pi}{\sqrt{3}}\Big(\sum_{n\neq 0} |\hat{f'}_n|^2\Big)^{1/2} < \infty \quad ///$$

<u>Remark</u>: This sufficient cond. is not necessarily the sharpest one. We won't prove the following sharper thm's:

<u>Thm</u> (Bernstein, 1914) $f \in Lip_\alpha(\mathbb{R}) = C^\alpha(\mathbb{R})$, $\alpha > \frac{1}{2}$

$\Rightarrow$ The F.S. of f conv. abs. & unif.

<u>Thm</u> (Zygmund, 1928) $f \in Lip_\alpha(\mathbb{R}) \cap BV(\mathbb{R})$, $\alpha > 0$

$\Rightarrow$ The F.S. of f conv. abs. & unif.

Thm (Smoothness class & Fourier coef)
Suppose f : 2π-periodic, $\in C^{k-1}(\mathbb{R})$, and $f^{(k-1)} \in PS(\mathbb{R})$, i.e., $f^{(k)}$ exists except perhaps at finitely many pts in each bdd. interval.
Then, $\sum_{-\infty}^{\infty} |n^k \hat{f}_n|^2 < \infty$.
In particular, $n^k \hat{f}_n \to 0$ as $n \to \pm\infty$
On the other hand, suppose the Fourier coef's satisfy $|\hat{f}_n| \leq C|n|^{-(k+\alpha)}$ $\exists c>0$, $\exists \alpha > 1$ for $n \in \mathbb{Z} \setminus \{0\}$. Then $f \in C^k(\mathbb{R})$.

(Proof) The first part: Apply the Deriv. Thm ($\hat{f'_n} = in\hat{f}_n$) k times to get $\widehat{f^{(k)}_n} = (in)^k \hat{f}_n$.

Apply Bessel's ineq. to $f^{(k)}$ to get

$$\sum_{-\infty}^{\infty} |n^k \hat{f}_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f^{(k)}(\theta)|^2 d\theta < \infty \quad //$$

The second part:

$$\sum_{n\neq 0} |n^j \hat{f}_n| \leq C \sum_{n \neq 0} |n|^{-(k-j+\alpha)} \leq 2C \sum_{n>0} n^{-\alpha} \quad \text{for } j \leq k$$

converges since $\alpha > 1$.

$\Rightarrow \sum (in)^j \hat{f}_n e^{in\theta}$ conv. abs. & unif. to $f^{(j)}$, $j \leq k$.
$\Rightarrow f^{(j)} \in C(\mathbb{R})$, $j \leq k$. $\Rightarrow f \in C^k(\mathbb{R})$. ///