Lecture 16 : Fourier Series V

Note Title

☆ Fourier Series on Intervals II

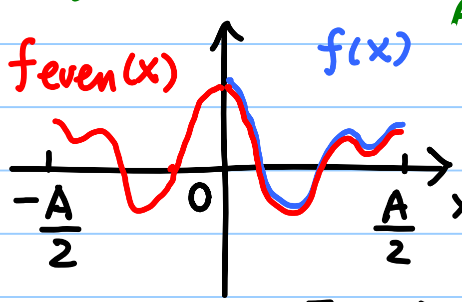
Suppose your fcn is defined on $[0, \frac{A}{2}]$ instead of $[-\frac{A}{2}, \frac{A}{2}]$.

$\Rightarrow \exists$ two ways to make it an A-periodic fcn:

(1) Even extension

$$f_{even}(x) = \begin{cases} f(x) & \text{if } x \in [0, \frac{A}{2}] \\ f(-x) & \text{if } x \in [-\frac{A}{2}, 0] \end{cases}$$

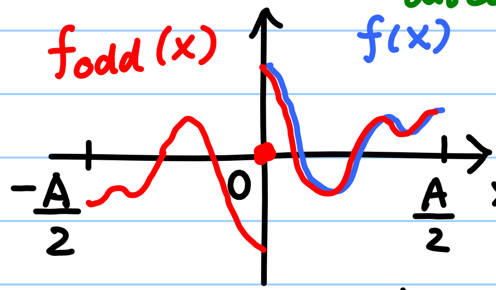
$f \in C[0, \frac{A}{2}] \Rightarrow f_{even} \in C(\mathbb{R})$
A-periodic



(2) Odd extension

$$f_{odd}(x) = \begin{cases} f(x) & \text{if } x \in (0, \frac{A}{2}] \\ 0 & \text{if } x = 0 \\ -f(-x) & \text{if } x \in [-\frac{A}{2}, 0) \end{cases}$$

$f \in C[0, A/2] \Rightarrow f_{odd}$: discontinuous



Then their Fourier series expansions get simpler. But before computing them, let's review the relationship between the Fourier coefficients $\{\alpha_k\}$ w.r.t. the ONB $\{\frac{1}{\sqrt{A}} e^{2\pi i k x/A}\}$ and $\{a_k, b_k\}$ w.r.t. the ONB $\{\frac{1}{\sqrt{A}}\} \cup \{\sqrt{\frac{2}{A}} \cos(\frac{2\pi k}{A} x)\} \cup \{\sqrt{\frac{2}{A}} \sin(\frac{2\pi k}{A} x)\}$.

Let $g(x)$ be an A-periodic L^2 fcn.

$$g(x) \sim \frac{1}{\sqrt{A}} \sum_{k \in \mathbb{Z}} \alpha_k e^{2\pi i k x/A}, \quad \alpha_k = \frac{1}{\sqrt{A}} \int_{-A/2}^{A/2} g(x) e^{-2\pi i k x/A} dx$$

Then $\frac{1}{\sqrt{A}} \sum_{k \in \mathbb{Z}} \alpha_k e^{2\pi i k x/A}$

$$= \frac{1}{\sqrt{A}} \left[\alpha_0 + \sum_{1}^{\infty} (\alpha_k + \alpha_{-k}) \cos\left(\frac{2\pi k x}{A}\right) + i(\alpha_k - \alpha_{-k}) \sin\left(\frac{2\pi k x}{A}\right) \right]$$

$$= \frac{\alpha_0}{\sqrt{A}} + \sum_1^\infty \left[\underbrace{\frac{\alpha_k + \alpha_{-k}}{\sqrt{2}}}_{a_k} \sqrt{\frac{2}{A}} \cos\left(\frac{2\pi k x}{A}\right) + \underbrace{\frac{\alpha_k - \alpha_{-k}}{\sqrt{2}} i}_{b_k} \sqrt{\frac{2}{A}} \sin\left(\frac{2\pi k x}{A}\right) \right]$$

If we want, we can write a_0 instead of α_0.

<u>Remark</u>: In many books, the Fourier series is often defined on $[-\pi, \pi)$ and written as

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

where $a_0 = 2c_0$, $\underset{k \geq 1}{a_k = c_k + c_{-k}}$, $b_k = i(c_k - c_{-k})$.

But $\{e^{ikx}\}$ is an orthogonal basis of $L^2[-\pi, \pi]$, but not normalized. To make it an ONB, one needs the factor $\frac{1}{\sqrt{2\pi}}$.

The same can be said for the orthogonal basis $\{1\} \cup \{\cos kx\} \cup \{\sin kx\}$.
The ONB is $\left\{\frac{1}{\sqrt{2\pi}}\right\} \cup \left\{\frac{1}{\sqrt{\pi}} \cos kx\right\} \cup \left\{\frac{1}{\sqrt{\pi}} \sin kx\right\}$

i.e., $A = 2\pi$.

Compare this notation with mine in this lecture, which is the orthonormalized version:

$$\sum_{k=-\infty}^{\infty} \alpha_k \frac{1}{\sqrt{A}} e^{2\pi i k x/A} = \frac{1}{\sqrt{A}} a_0 + \sqrt{\frac{2}{A}} \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{2\pi k}{A} x\right) + b_k \sin\left(\frac{2\pi k}{A} x\right) \right]$$

where $a_0 = \alpha_0$, $a_k = \frac{\alpha_k + \alpha_{-k}}{\sqrt{2}}$, $b_k = \frac{\alpha_k - \alpha_{-k}}{\sqrt{2}} i$, $k \geq 1$.

Now, let's go back to the Fourier series expansion of f_{even} & f_{odd}.

$$\alpha_k[f_{even}] = \frac{1}{\sqrt{A}} \int_{-A/2}^{A/2} f_{even}(x)\, e^{-2\pi i k x/A}\, dx$$
$$= \frac{2}{\sqrt{A}} \int_0^{A/2} f(x) \cos\left(\frac{2\pi k x}{A}\right) dx$$
$$= \alpha_{-k}[f_{even}]$$
thanks to the evenness of $\cos\theta$.

Recall the relationship:

$$a_0 = \alpha_0, \quad a_k = \frac{\alpha_k + \alpha_{-k}}{\sqrt{2}}, \quad b_k = \frac{\alpha_k - \alpha_{-k}}{\sqrt{2}};$$

Hence, in this case of f_{even}, $a_0 = \alpha_0[f_{even}]$, $a_k = \sqrt{2}\,\alpha_k[f_{even}]$, $b_k \equiv 0$, $k \geq 1$.

In other words, f_{even} can be written as the Fourier cosine series:

$$f_{even}(x) \sim \frac{1}{\sqrt{A}} a_0 + \sum_{k=1}^{\infty} a_k \sqrt{\frac{2}{A}} \cos\left(\frac{2\pi k x}{A}\right)$$

where
$$a_0 = \frac{2}{\sqrt{A}} \int_0^{A/2} f(x)\, dx = \alpha_0[f_{even}],$$
$$a_k = 2\sqrt{\frac{2}{A}} \int_0^{A/2} f(x) \cos\left(\frac{2\pi k x}{A}\right) dx$$
$$= \sqrt{2}\,\alpha_k[f_{even}].$$

Similarly, for $f_{odd}(x)$,

$$\alpha_k[f_{odd}] = \frac{1}{\sqrt{A}} \int_{-A/2}^{A/2} f_{odd}(x) e^{-2\pi i k x/A} dx$$

$$= \frac{-2i}{\sqrt{A}} \int_0^{A/2} f(x) \sin\left(\frac{2k\pi x}{A}\right) dx$$

$$= -\alpha_{-k}[f_{odd}]$$ due to the oddness of $\sin\theta$.

$$\Rightarrow \quad a_k = \frac{\alpha_k + \alpha_{-k}}{\sqrt{2}} \equiv 0, \quad k \in \mathbb{N}.$$

$$a_0 = \alpha_0 = \frac{1}{\sqrt{A}} \int_{-A/2}^{A/2} f_{odd}(x)\, dx = 0.$$

No cosines

$$b_k = \frac{\alpha_k - \alpha_{-k}}{\sqrt{2}} i = \sqrt{2}\, i\, \alpha_k[f_{odd}].$$

So, $$f_{odd}(x) \sim \sum_{k=1}^{\infty} b_k \sqrt{\frac{2}{A}} \sin\left(\frac{2\pi k x}{A}\right)$$

$$b_k = 2\sqrt{\frac{2}{A}} \int_0^{A/2} f(x) \sin\left(\frac{2\pi k x}{A}\right) dx.$$

So, if a fcn is given on $[0, \frac{A}{2}]$, say $f \in C[0, A/2]$
$\exists$ three ways to extend it to a periodic fcn

$O(1/k)$ (1) Brute force periodization with period $A/2$.

$O(1/k^2)$ (2) Even extension followed by A-periodization.

$O(1/k)$ (3) Odd 〃 〃 〃

(2) is the best among these 3 in terms of the decay of the Fourier coeff's.

However, $\exists$ an even better way!

☆ The Lanczos Method (1938)

Suppose $f \in C^{2m}[0,1]$, but $f(0) \neq f(1)$ no match at $x=0,1$.
also assume $f^{(2m+1)} \in BV[0,1]$. $m = 1, 2, \cdots$.

Lanczos's idea:
decompose $f(x) = u(x) + v(x)$

where $u(x)$ = a polynomial of degree $2m-1$.
s.t. $\begin{cases} u^{(2k)}(0) = f^{(2k)}(0) \\ u^{(2k)}(1) = f^{(2k)}(1) \end{cases}$, $k = 0, 1, \cdots, m-1$.

Then, consider the odd extension of v.

$\Rightarrow$ $v \in C^{2m-1}(\mathbb{R})$, $v^{(k)}(0) = v^{(k)}(1) = 0$
$k = 0, 1, 2, \cdots, 2m-1$.

and the Fourier sine coefficients of $v(x)$ (with period 2) decay as

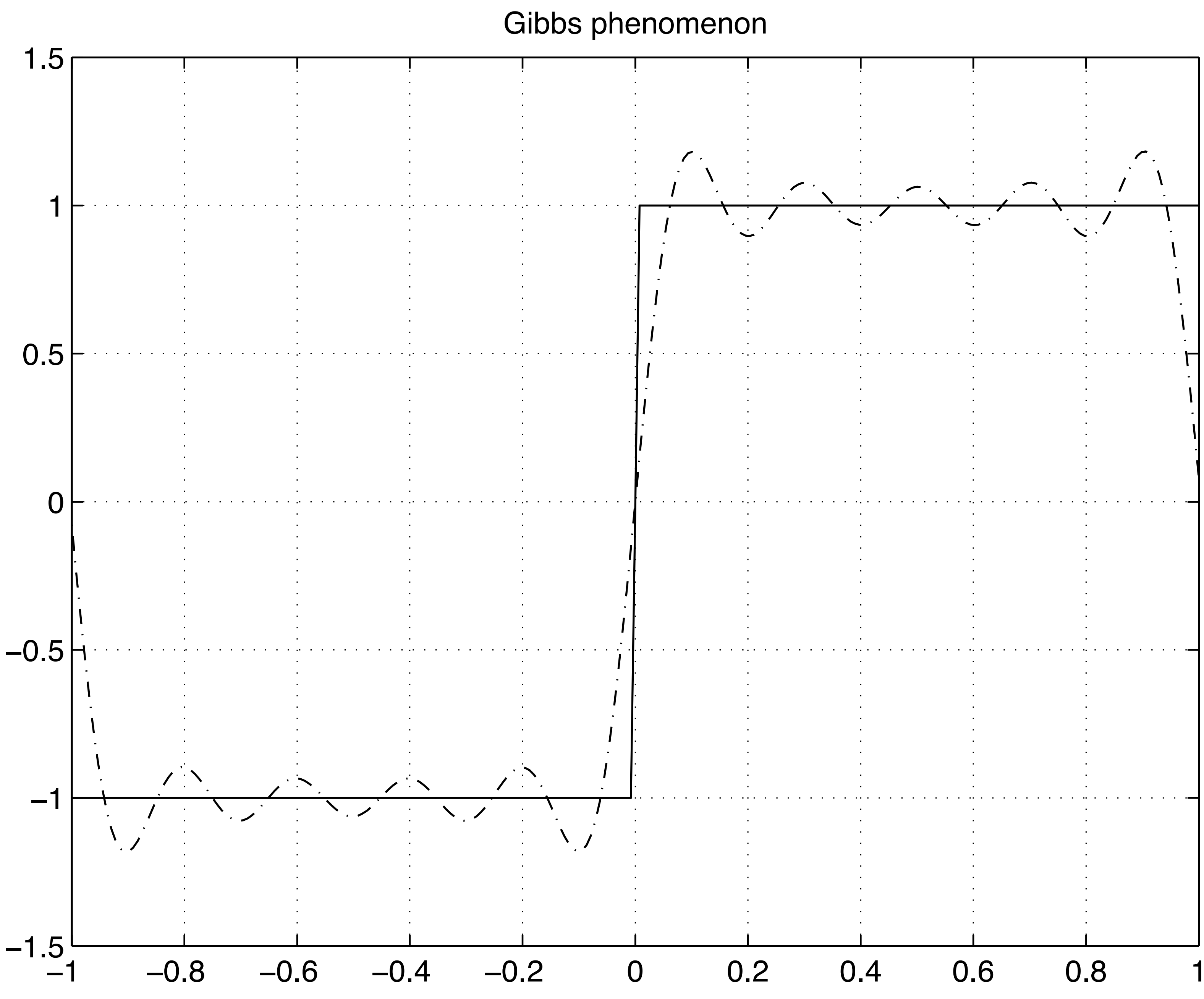
$$b_k = O(|k|^{-2m-1})$$

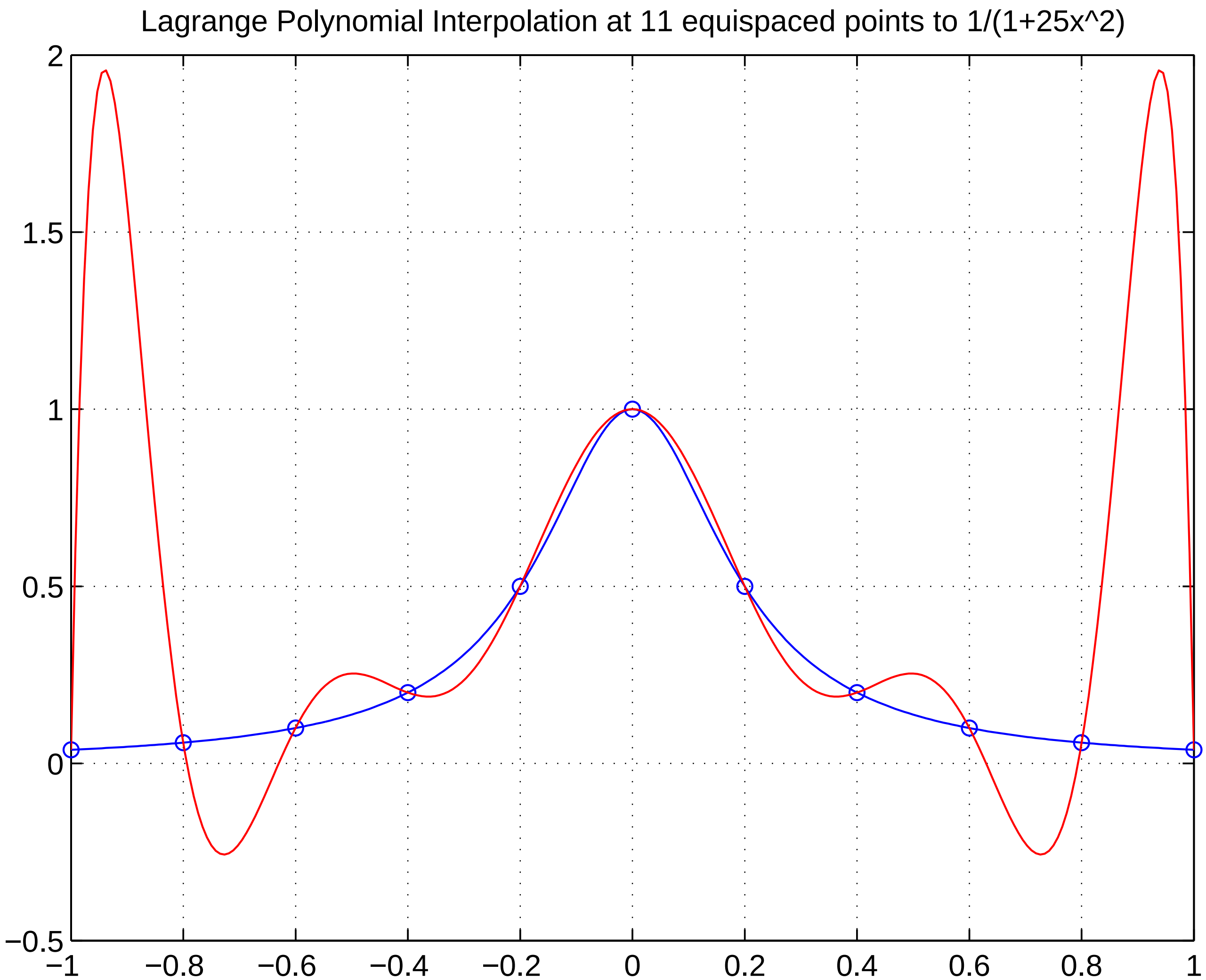
e.g., $m=1$ gives us $b_k = O(1/k^3)$.
and $u(x)$ is a straight line connecting $(0, f(0))$ & $(1, f(1))$.

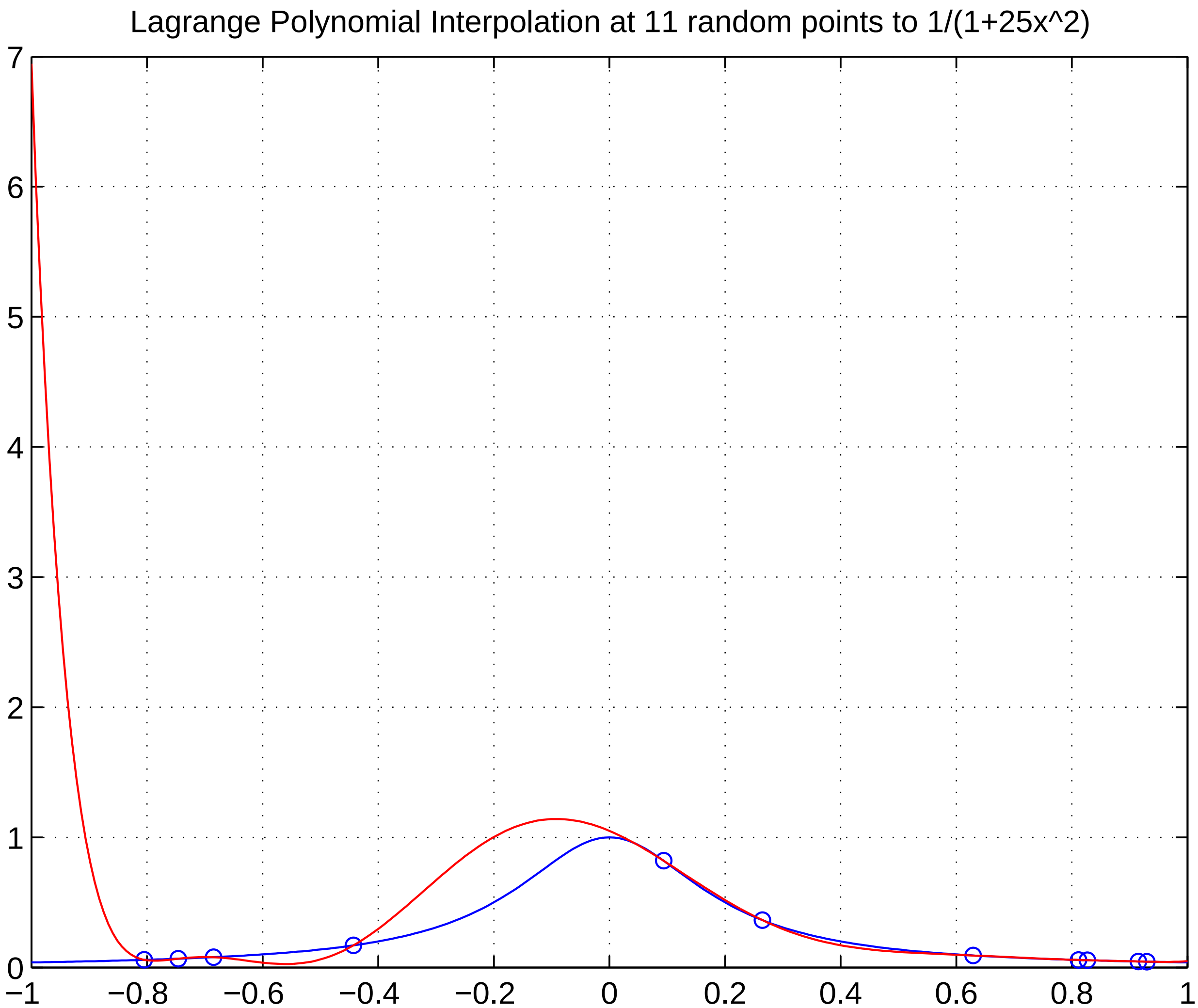
Remarks:

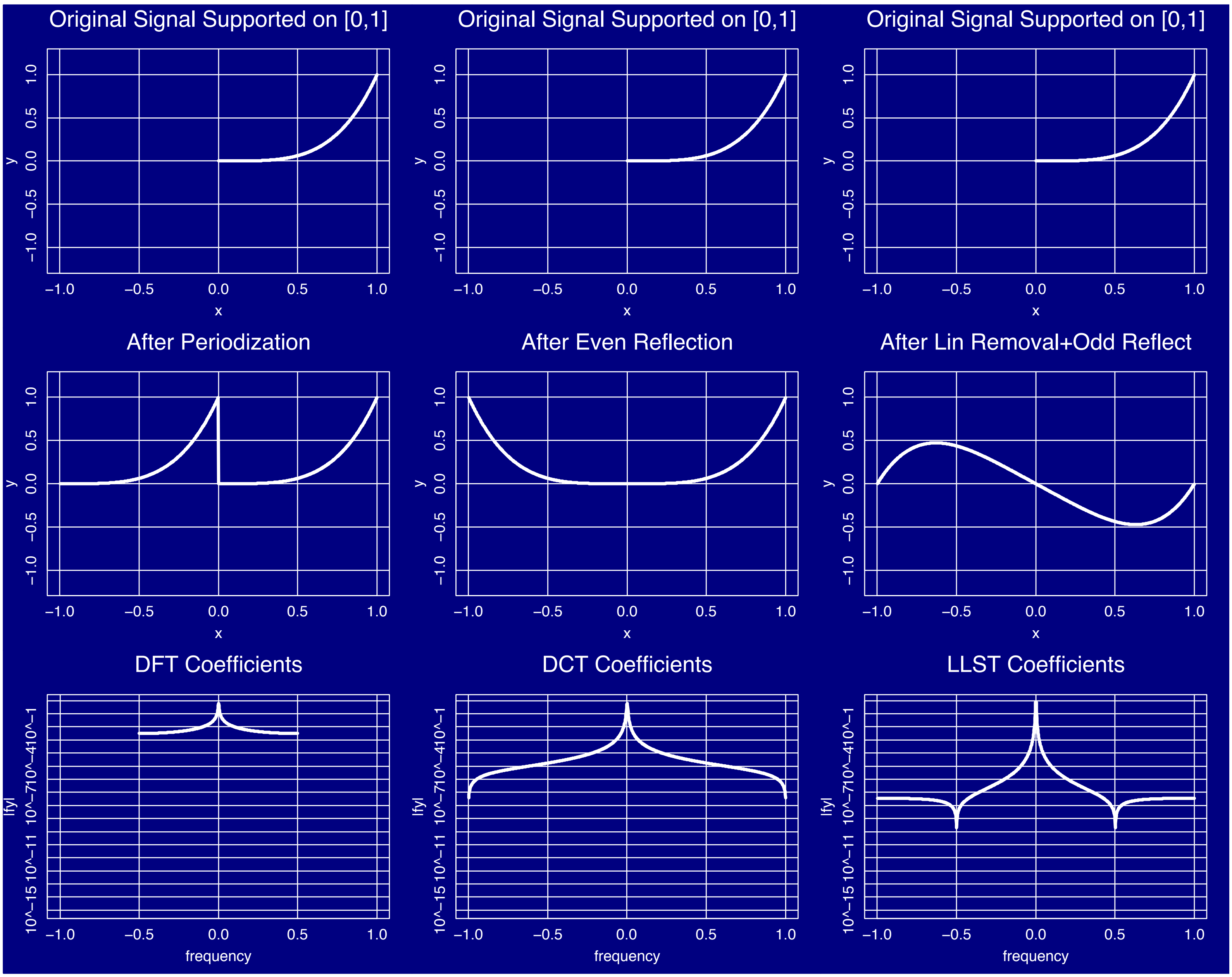
(1) $f = u + v$ = (an algebraic poly) + (a trig. poly).
can avoid both the Runge & Gibbs phenomena!

(2) NS–J.F. Remy (2006) generalized this to $\mathbb{R}^d$.


Gibbs phenomenon
1.5
1
0.5
0
–0.5
–1
–1.5
–1
–0.8
–0.6
–0.4
–0.2
0
0.2
0.4
0.6
0.8
1


Lagrange Polynomial Interpolation at 11 equispaced points to 1/(1+25x^2)
2
1.5
1
0.5
0
−0.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


Lagrange Polynomial Interpolation at 11 random points to 1/(1+25x^2)
7
6
5
4
3
2
1
0
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


Original Signal Supported on [0,1]
Original Signal Supported on [0,1]
Original Signal Supported on [0,1]
After Periodization
After Even Reflection
After Lin Removal+Odd Reflect
DFT Coefficients
DCT Coefficients
LLST Coefficients
x
y
frequency
|fy|
−1.0 −0.5 0.0 0.5 1.0
10^−15 10^−11 10^−7 10^−4 10^−1