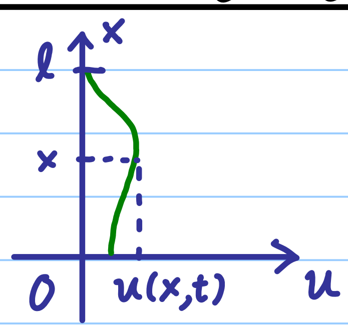
Lecture 20: Sturm-Liouville Systems I

Note Title

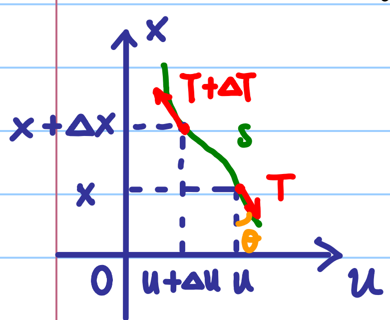
- We'll discuss the generalization of the Fourier series (instead of cosines, sines, and $\mathbb{C}$-exp.).
- We'll justify the method of separation of variables.
- We'll use an example of a hanging chain (Daniel Bernoulli 1738; Euler 1784). Other common examples include heat, wave, potential eqn's on a 2D disk, a 3D ball, etc. (Some of them will be assigned as HW).

☆ Small oscillations of a hanging chain



- a uniform heavy flexible chain of length l is freely suspended at one end and hangs under gravity.
- The chain is displaced slightly in a vertical plane and released from rest.

- assume that movement only occurs in this vertical plane (x,u) where $u(x,t)$ is the horizontal displacement of this chain at height x and time t.
- The tension T in the chain at height $x \approx$ the weight beneath that point $\approx \rho g x$, ρ: density of the chain.



- The horizontal component of tension at x is $T\sin\theta \approx \rho g x \frac{\partial u}{\partial x}(x)$ (recall Lecture 7!)
- Similarly at $x+\Delta x$,

$$T+\Delta T\Big|_{hor} \approx \rho g (x+\Delta x)\frac{\partial u}{\partial x}(x+\Delta x)$$
$$\approx \rho g(x+\Delta x)\left(\frac{\partial u}{\partial x}(x) + \Delta x \frac{\partial^2 u}{\partial x^2}(\xi)\right)$$

$$\Rightarrow \Delta T\Big|_{hor} \approx \rho g \Delta x\left(\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2}\right) = \rho g \Delta x \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)$$

So, the eqn. of motion is

$$\rho \Delta x \frac{\partial^2 u}{\partial t^2} = \rho g \Delta x \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) \Rightarrow \frac{\partial^2 u}{\partial t^2} = g\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right).$$

By scaling the t variable $t \to \sqrt{g}\, t$, we get

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(x \frac{\partial u}{\partial x}\right)$$

$$\begin{cases} \text{B.C.}: & u(\ell,t)=0, \quad \forall t \geq 0. \\ \text{I.C.}: & u(x,0) = u_0(x) \quad \forall x \in [0,\ell]. \\ & u_t(x,0) = 0 \\ \text{Bddness}: & \sup |u(x,t)| < \infty, \quad \forall x \in [0,\ell], \ \forall t \geq 0. \end{cases}$$

Again via separation of variables, i.e., $u(x,t) = f(x)g(t)$ (this form satisfying B.C. & BDD. but not I.C. is called the normal mode of the system), we get

$$f(x)\, g''(t) = g(t)\left(f'(x) + x f''(x)\right)$$

$$\Rightarrow \frac{g''(t)}{g(t)} = \frac{f'(x)}{f(x)} + x\frac{f''(x)}{f(x)} \quad \text{whenever } f(x)g(t) \neq 0.$$

So, must be $= -\lambda$ (const.)

$$\Leftrightarrow \begin{cases} g'' + \lambda g = 0, \quad \forall t \geq 0 \\ (*)\ (x f')' + \lambda f = 0, \quad \forall x \in [0,\ell], \ f(\ell) = 0. \end{cases}$$

Similarly as before, $g(t) = a \cos\sqrt{\lambda}\, t + b \sin\sqrt{\lambda}\, t$

$\lambda > 0$, otherwise g hence u becomes unbdd.

But now, how to solve $(*)$?

Let's change $x \to \xi = 2\sqrt{\lambda x}$

$f(x) \to \tilde{f}(\xi)$

Then, $$\frac{d^2 \tilde{f}}{d\xi^2} + \frac{1}{\xi}\frac{d\tilde{f}}{d\xi} + \tilde{f} = 0 \quad \text{——} \ (**)$$

This is called Bessel's eqn. of order 0.

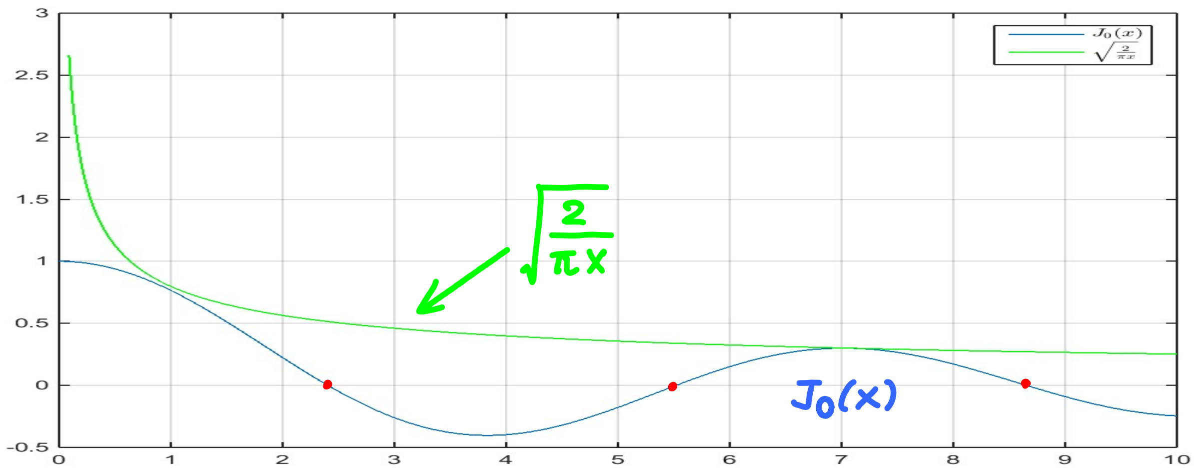
<u>Remark</u>: In general, Bessel's eqn. of order ν is defined as $x^2 f'' + x f' + (x^2 - \nu^2) f = 0$.

The general sol. of (**) is

$$\hat{f}(\xi) = c\,\underbrace{J_0(\xi)}, \quad c: \text{arb. const.}$$

The Bessel fcn (of the first kind) of order 0. For $x \geq 1$, the following holds:

$$J_0(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right) + E_0(x) \text{ with } |E_0(x)| \leq C x^{-3/2}.$$



So, $f(x) = c\,J_0(2\sqrt{\lambda x})$

Now, due to the B.C., we must have

$f(\ell) = c\,J_0(2\sqrt{\lambda \ell}) = 0$, i.e.,

$2\sqrt{\lambda \ell}$ = zeros of $J_0(\xi)$!

↪ tabulated extensively in many books & software systems, e.g., MATLAB, Mathematica

Let j_k be the kth zero of $J_0(\xi)$, $k = 1, 2, \cdots$

$j_1 \approx 2.40483$, $j_2 \approx 5.52008$, $j_3 \approx 8.65373$, $\cdots$

So, $2\sqrt{\lambda_k \ell} = j_k \iff \lambda_k = \dfrac{j_k^2}{4\ell}$

Hence, $f(x) = f_k(x) = c_k\,J_0\left(j_k \sqrt{\frac{x}{\ell}}\right)$

$g(t) = g_k(t) = a_k \cos \dfrac{j_k t}{2\sqrt{\ell}} + b_k \sin \dfrac{j_k t}{2\sqrt{\ell}}$.

$$\Rightarrow u_k(x,t) = J_0\left(j_k\sqrt{\frac{x}{\ell}}\right)\left(a_k \cos\frac{j_k t}{2\sqrt{\ell}} + b_k \sin\frac{j_k t}{2\sqrt{\ell}}\right).$$

c_k is absorbed in a_k, b_k.

By the superposition principle,

$$u(x,t) = \sum_{k=1}^{\infty} J_0\left(j_k\sqrt{\frac{x}{\ell}}\right)\left(a_k \cos\frac{j_k t}{2\sqrt{\ell}} + b_k \sin\frac{j_k t}{2\sqrt{\ell}}\right).$$

By the I.C.: $u(x,0) = \sum_{k=1}^{\infty} a_k J_0\left(j_k\sqrt{\frac{x}{\ell}}\right) = u_0(x)$

$\forall x \in [0,\ell]$ $\quad u_t(x,0) = \sum_{k=1}^{\infty} \frac{b_k j_k}{2\sqrt{\ell}} J_0\left(j_k\sqrt{\frac{x}{\ell}}\right) \equiv 0$

$$\Rightarrow b_k \equiv 0,\ \forall k \in \mathbb{N}.$$

$\{a_k\}$ is the (generalized) Fourier coeff's of $u_0(x)$ w.r.t. an appropriate ONB of $L^2[0,\ell]$ associated with $\left\{J_0\left(j_k\sqrt{\frac{x}{\ell}}\right)\right\}_1^{\infty}$.

Note $\{J_0(j_k x/\ell)\}_1^{\infty}$ form an OB of $L^2_w[0,\ell]$ with $w=x$.

All the previous questions (and answers) for $L^2[-\pi,\pi]$ or $L^2[0,\pi]$ with sines, cosines, and $\mathbb{C}$-exp. emerge again!

- We reached the sol. of the hanging chain problem with the Bessel fcns, but if $\rho=\rho(x)$, it's not clear whether we can obtain a series sol. in a similar manner.
- Whether we can do so was answered by Sturm & Liouville in 1830's. Around that period, the theory of Hilbert spaces & self-adjoint operators were not yet invented.
- However, we'll approach this problem through the rudimentary Hilbert space theory & self-adj. op's due to its power and the viewpoint they provide!

<u>Def</u>. A **regular Sturm-Liouville system** consists of a differential eqn. on a *finite* interval $[a,b] \subset \mathbb{R}$ with the B.C. as follows:

$$\text{RSL}: \begin{cases} (pf')' + (\lambda w + q) f = 0 \\ \alpha f(a) + \alpha' f'(a) = 0 \\ \beta f(b) + \beta' f'(b) = 0 \end{cases}$$ separated homog. Robin B.C.

where $p \in C'[a,b]$, $q, w \in C[a,b]$, all real-valued, $p > 0$, $w > 0$ on $[a,b]$, $\alpha, \alpha', \beta, \beta' \in \mathbb{R}$ and the cases $\alpha = \alpha' = 0$ and $\beta = \beta' = 0$ are excluded.

Ex. In the H.C. problem, $p(x) = x$, $q(x) = 0$, $w(x) = 1$. $a = 0$, $b = l$, $\beta' = 0$, $\beta \neq 0$.

But unfortunately, the H.C. problem is not an RSL prob. since $\alpha = \alpha' = 0$ (i.e., no B.C. at $x = 0$).

Def. A *singular Sturm-Liouville system* consists of a D.E. on a half or open interval together with the following type of B.C.:

$$\text{SSL}: \begin{cases} (pf')' + (\lambda w + q) f = 0 & \text{on } I = (a,b], [a,b), \text{ or } (a,b) \\ \beta f(b) + \beta' f'(b) = 0 & \text{if } I = (a,b] \\ \alpha f(a) + \alpha' f'(a) = 0 & \text{if } I = [a,b) \\ f: \text{bdd on } I \text{ but no B.C.} & \text{if } I = (a,b) \end{cases}$$

where $p \in C'[a,b]$, $q, w \in C[a,b]$

$p > 0$ on I, $w > 0$ on $[a,b]$

$p(a) = 0$ if $I = (a,b]$

$p(b) = 0$ if $I = [a,b)$

$p(a) = p(b) = 0$ if $I = (a,b)$.

The DEs in RSL/SSL may look very special and rather limited to you, but that's not the case. One can convert a rather general DE to this form.

Ex. $x f'' + (1+3x) f' + x^2 f = 0$

$\iff (x e^{3x} f')' + x^2 e^{3x} f = 0$, i.e., $p = x e^{3x}$, $q = x^2 e^{3x}$, $w = 0$.