Lecture 21: Sturm-Liouville Systems II

Note Title

☆ Eigenfcns & Eigenvalues

<u>Def</u>. An eigenfcn of an RSL/SSL system corresponding to a scalar λ is a non-zero C^2 fcn satisfying the DE & BC in the SL system.

$$(pf')' + (\lambda w + q)f = 0 \qquad \begin{cases} \alpha f(a) + \alpha' f'(a) = 0 \\ \beta f(b) + \beta' f'(b) = 0 \end{cases}$$

A scalar λ is said to be an eigenvalue of the system if $\exists$ an eigenfcn corresp. to λ.

We already looked at the RSL on $[0,\pi]$ $f'' + \lambda f = 0$ with $f(0) = f(\pi) = 0$ leading to the Fourier sine series where $\{\sin kx\}_1^\infty$ are the eigenfcns with $\lambda_k = k^2$.
There, the set up was: $p = 1$, $w = 1$, $q = 0$.

However, for more general p, w, q, it's difficult to impossible to find eigenfcns & eigenval's explicitly. (Except some well-known situations where special fcns of math. physics could be used; e.g., Bessel fcns, various orthogonal poly's including Legendre, Chebyshev, etc.)

Fourier did for the simplest cases (sines, cosines) in 1807 (published in 1822).
It's remarkable that Sturm & Liouville successfully worked out the general cases in 1837, a relatively short time since Fourier!
They showed $\exists$ enough # of eigenval's & eigenfcns so that the separation of variables are justified for a <u>wide range of physical problems</u>.

Before showing this, let's introduce some operator notation.

$$\mathcal{L} := \frac{d}{dx}\left(p\frac{d}{dx}\cdot\right) + q\cdot \quad ,$$ the differential operator of an SL system.

Let $D(\mathcal{L})$:= the domain of $\mathcal{L} = \{ f \in L^2(a,b) \mid f'' \in L^2(a,b),$ f : satisfies the B.C. $\}$.

The fcn space $\{ f \in L^2(a,b) \mid f'' \in L^2(a,b)\}$ (without the B.C.) is the so-called Sobolev space and denoted by $H^2(a,b)$ or $W^{2,2}(a,b)$.

Then, the RSL/SSL system can be simply written as $\mathcal{L}f + \lambda w f = 0$,

where $\mathcal{L} : D(\mathcal{L}) \longrightarrow L^2(a,b)$.

Note that $\mathcal{L}$ also specifies the B.C. via $D(\mathcal{L})$.

- <u>Lagrange's Identity</u>

For any $u, v \in D(\mathcal{L})$,

$$u\mathcal{L}v - v\mathcal{L}u = (p(uv' - vu'))'$$

(Proof) An easy exercise!

- <u>Review & Def. of an adjoint operator</u>

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a linear operator. Then its adjoint $A^* : \mathcal{H}_2 \to \mathcal{H}_1$ is defined by $\langle Au, v\rangle_{\mathcal{H}_2} = \langle u, A^*v\rangle_{\mathcal{H}_1}$, $\forall u \in \mathcal{H}_1, \forall v \in \mathcal{H}_2$.

<u>Ex.</u> $\mathcal{H}_1 = L^2(a,b)$, $\mathcal{H}_2 = L^2(c,d)$, $k(\cdot,\cdot) \in C([c,d]\times[a,b])$

$$Au := \int_a^b k(x,y)\,u(y)\,dy, \quad \forall u \in L^2(a,b).$$

$$\Rightarrow A^*v = \int_c^d \overline{k(y,x)}\,v(y)\,dy, \quad \forall v \in L^2(c,d).$$ ≈ matrix conj. transp.

- <u>Self-adjointness property</u>

For any $u, v \in D(\mathcal{L})$,
$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}v\rangle$
In other words, $\mathcal{L}^* = \mathcal{L}$ (implies $D(\mathcal{L}^*) = D(\mathcal{L})$).
$\langle \cdot, \cdot \rangle$ = the standard inner prod. in $L^2(a,b)$.

(Proof) Let $v \in D(\mathcal{L})$. Then $\bar{v} \in D(\mathcal{L})$ because $\alpha, \alpha', \beta, \beta' \in \mathbb{R}$ in the B.C.
Since p, q, w are also real-valued, $\overline{\mathcal{L}v} = \mathcal{L}\bar{v}$.

$$\Rightarrow \langle \mathcal{L}u, v\rangle - \langle u, \mathcal{L}v\rangle = \int_a^b (\bar{v}\mathcal{L}u - \overline{\mathcal{L}v}\,u)\,dx$$
$$= \int_a^b (\bar{v}\mathcal{L}u - u\mathcal{L}\bar{v})\,dx$$
$$\text{Lagrange's Id.} \rightarrow \quad = \big[p(u\bar{v}' - \bar{v}u')\big]_a^b$$

which turns out to be 0 regardless of RSL/SSL thanks to the B.C. or the end point cond. of p.
For example, at $x=a$, if $p(a)=0$, it's obvious.
<u>If</u> $p(a)>0$, <u>then</u> since both $u, \bar{v} \in D(\mathcal{L})$, i.e., satisfy the B.C.,
$$\begin{bmatrix} u(a) & u'(a) \\ \bar{v}(a) & \bar{v}'(a) \end{bmatrix} \begin{bmatrix} \alpha \\ \alpha' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
where $\alpha^2 + \alpha'^2 \neq 0$. So
$$\begin{vmatrix} u(a) & u'(a) \\ \bar{v}(a) & \bar{v}'(a) \end{vmatrix} = u(a)\bar{v}'(a) - \bar{v}(a)u'(a) = 0$$
is a must! At $x=b$, we do similarly. ///

<u>Thm</u> The eigenval's of an SL system are reals.

(Proof) Let λ be an eigenval of an SL system.
Let f be the corresp. eigenfcn, i.e.,
$\mathcal{L}f = -\lambda w f$, $f \not\equiv 0$.

Because $\mathcal{L}$ is self-adjoint, we have

$$\begin{aligned} 0 &= \langle \mathcal{L}f, f \rangle - \langle f, \mathcal{L}f \rangle \\ &= \langle -\lambda w f, f \rangle - \langle f, -\lambda w f \rangle \\ &= -\lambda \langle wf, f \rangle + \bar{\lambda} \langle f, wf \rangle \\ &= (\bar{\lambda} - \lambda) \int_a^b w(x) |f(x)|^2 dx \end{aligned}$$

$\Rightarrow$ $\bar{\lambda} = \lambda$ is a must since $w > 0$, $f \not\equiv 0$ on $[a, b]$.

$\Leftrightarrow$ $\lambda \in \mathbb{R}$. ///

☆ Orthogonality of Eigenfcns

Thm Let u, v be eigenfcns of an SL system corresp. to the eigenval's λ, μ, respectively with $\lambda \neq \mu$. Then $\sqrt{w}\, u \perp \sqrt{w}\, v$ (or $u \perp v$ in $\langle \cdot, \cdot \rangle_w$).

(Proof) By the assumption, $\mathcal{L}u = -\lambda w u$, $\mathcal{L}v = -\mu w v$

By the self-adj'ness, we have $\lambda, \mu \in \mathbb{R}$.

$$\begin{aligned} 0 &= \langle \mathcal{L}u, v \rangle - \langle u, \mathcal{L}v \rangle \\ &= \langle -\lambda w u, v \rangle - \langle u, -\mu w v \rangle \\ &= (\mu - \lambda) \int_a^b w(x) u(x) \overline{v(x)}\, dx \\ &= \underbrace{(\mu - \lambda)}_{\neq 0} \underbrace{\langle \sqrt{w}\, u, \sqrt{w}\, v \rangle}_{=0} \end{aligned}$$

$\neq 0 \Rightarrow$ $= 0$ is a must. ///

- We are approaching now diagonalization of the diff. op. $\mathcal{L}$ using the eigenfcns. By normalizing each eigenfcn, we get an ON set in $L^2(a,b)$. Then the question is: Does this set form an ONB for $L^2(a,b)$, i.e., is this a complete system?

• At this point, however, we even do not know if $\exists$ enough eigenfcns for a given SL system. In fact, at this moment, we even do not know if $\exists$ any eigenval/eigenfcn for the SL system. (More about this when we discuss the "eigenfcn expansion".)
Here, we can show $\exists$ not too many eigval's.

<u>Thm</u> Not every real number is an eigenval of RSL (i.e., the set of eigenval's of RSL is countable). The cardinality of λ's $= \aleph_0 < c =$ the cardinality of continuum.
alef 0

(Proof) Facts to use: ① The union of countable seq. of countable sets is countable; ② $\mathbb{R}$ is uncountable.
Let $\{e_n\}$ be an ONB in $L^2(a,b)$. We know $\exists$ such an ONB, e.g., $e_n(x) = (b-a)^{-1/2} e^{2\pi i n(x-\frac{a+b}{2})/(b-a)}$.
Now, suppose that every $\lambda \in \mathbb{R}$ is an eigenval of an RSL system. By the orthogonality thm, $\exists$ an ON set $\{f_\lambda\}_{\lambda \in \mathbb{R}}$ in $L^2(a,b)$.
For each $n \in \mathbb{N}$, let $E_n := \{\lambda \in \mathbb{R} \mid \langle e_n, f_\lambda \rangle \neq 0\}$.
Define also $E_n^m := \{\lambda \in \mathbb{R} \mid |\langle e_n, f_\lambda\rangle| \geq 1/m\}$.
Then $E_n = \bigcup_{m=1}^{\infty} E_n^m$. By Bessel's ineq., we know for $\forall g \in L^2(a,b)$, the set $\{\lambda \in \mathbb{R} \mid |\langle g, f_\lambda\rangle| \geq c\}$ is finite for any $c > 0$ because $\sum |\langle g, f_\lambda\rangle|^2 \leq \|g\|^2 < \infty$.
So, by ①, E_n is countable for each $n \in \mathbb{N}$.
$\Rightarrow \bigcup_{n\in\mathbb{N}} E_n$ is a countable subset of $\mathbb{R}$, so by ②, $\bigcup_1^\infty E_n$ is a proper subset of $\mathbb{R}$.
Now take any $\lambda \in \mathbb{R} \setminus \bigcup_1^\infty E_n$.

Then for such λ, $f_\lambda \perp e_n$, $\forall n \in \mathbb{N}$.
But $f_\lambda \not\equiv 0$ by def.
$\Rightarrow$ Contradicting the completeness of $\{e_n\}_1^\infty$ as an ONB of $L^2(a, b)$ #

<u>Thm</u> Consider the RSL system with nonseparable self-adjoint B.C.:

$$B_j(f) = \alpha_j f(a) + \alpha_j' f'(a) + \beta_j f(b) + \beta_j' f'(b) = 0, \; j=1,2.$$

where $\alpha_j, \alpha_j', \beta_j, \beta_j' \in \mathbb{R}$. (Note this includes all the prev. cases)
Then, the eigenspace for any eigenval. λ is at most 2 dimensional (i.e., the multiplicity of $\lambda \leq 2$).
If the B.C.'s are separated (as in the original RSL), it's always 1-dimensional, i.e., each eigenval. λ is simple.

(Proof) The Fundamental Existence Thm for 2nd order ODEs states that for any const's c_1, c_2, $\exists!$ a sol. of $\mathcal{L}f + \lambda w f = 0$ satisfying the I.C.: $f(a) = c_1$, $f'(a) = c_2$.
$\Rightarrow$ a sol. is specified by two arb. param's, i.e., the space of all sol's (w. or w.o. satisfying the B.C.'s) of $\mathcal{L}f + \lambda w f = 0$ is 2D. So, the space of sol's satisfying the given B.C.'s is at most 2D. Moreover, if the B.C.'s are separated, one of them has a form $\alpha f(a) + \alpha' f'(a) = 0$ $\Longleftrightarrow$ c_1, c_2 are now constrained by $\alpha c_1 + \alpha' c_2 = 0$.
So, this reduces the dim. by 1. ///

<u>Remark</u>: If we impose one more B.C., $\beta f(b) + \beta' f'(b) = 0$, <u>then</u> it also reduces the dim. by 1. This could lead to the dim. of the sol. space $= 0$. But that is for general val. of λ. This is why $\exists$ nontrivial sol's only for certain special val's of λ, i.e., eigenval's!!

Thm (The Fund. Existence Thm for 2nd Order ODEs)
Let $P, Q, R \in C[a,b]$, real-valued. Let $x_0 \in [a,b]$, $y_0, y_1 \in \mathbb{R}$. Then, the initial value problem (IVP)

$$\begin{cases} y'' + P(x)y' + Q(x)y = R(x) & \text{on } [a,b] \\ y(x_0) = y_0, \quad y'(x_0) = y_1 \end{cases}$$

has a unique sol. on $[a,b]$, which is real-valued.

(Proof) See e.g., Appendix III B of the book of Sagan, our textbook, or any other standard books on ODEs. ///

Remark: Of course, this thm is applicable to the RSL systems since

$$(pf')' + qf = 0 \iff pf'' + p'f' + qf = 0$$

$$p > 0 \text{ on } [a,b], \text{ so } f'' + \underbrace{\frac{p'}{p}}_{=P} f' + \underbrace{\frac{q}{p}}_{=Q} f = 0$$

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