Lecture 24: Green's Functions III

Note Title

☆ Toward the Spectral Thm

Thm Let T be a self-adj. op on $\mathcal{H}$.
Then all eigval's of T are real, and eigfcns of T corresp. to distinct eigval's are orthogonal.
(We already did this for an SL system assuming that eig pairs exist. The proof is the same.)

Thm Let K be a compact self-adj. op. on $\mathcal{H}$.
Then, $\sigma(K) := \{\text{the set of eigval's of } K\} \subset \mathbb{R}$, which is either finite or a countable seq. tending to 0.

(Proof) Suppose K has infinitely many eigval's that are not tending to 0. Then $\exists\, \varepsilon > 0$ s.t. $\#\{n \in \mathbb{N} \mid |\lambda_n| > \varepsilon\} = +\infty$. Pick a (sub) seq. $\{\mu_n\}_{n=1}^{\infty} \subset \{\lambda_n\}_{n=1}^{\infty}$ and $|\mu_n| > \varepsilon$ $\forall n \in \mathbb{N}$ s.t. $\mu_n \neq \mu_m$ if $n \neq m$.
Let $K\varphi_n = \mu_n \varphi_n$, $\|\varphi_n\| = 1$. Then such $\{\varphi_n\}_1^{\infty}$ form an ON seq. (set) and
$\|K\varphi_n - K\varphi_m\|^2 = \|\mu_n\varphi_n - \mu_m\varphi_m\|^2 = \mu_n^2 + \mu_m^2 > 2\varepsilon^2$
$\Rightarrow \{K\varphi_n\}_1^{\infty}$ has no Cauchy subseq., i.e., no convergent subseq. This contradicts K: compact. #

Thm (Spectral Thm) Let K be a compact self-adj. op. on $\mathcal{H}$. Then $\exists$ a finite or infinite ON seq. $\{\varphi_n\}$ of eigfcns of K with corresp. real eigval's $\{\lambda_n\}$ s.t. $Kx = \sum_n \lambda_n \langle x, \varphi_n \rangle \varphi_n$.
If $\{\lambda_n\}$ is an infinite seq., then $\lambda_n \to 0$ as $n \to \infty$.

Remark: at this point, $\{\varphi_n\}$ may or may not be an ONB.

In fact, if $\mathcal{H}$ is not separable, then no ON seq. is complete in $\mathcal{H}$. But most of $\mathcal{H}$ of our interest are separable. So, $\{\varphi_n\}$ can be extended to an ONB in a separable $\mathcal{H}$.

Def. A Hilbert space is separable if it admits a countable complete ON set, i.e., ONB.

- An example of non-separable Hilbert space:

Let $L := \{ f \in C(\mathbb{R}) \mid p(f) := \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(x)|^2 dx \right)^{1/2} < \infty \}$.

$p(\cdot)$ is just a seminorm, i.e., $p(f) = 0 \not\Rightarrow f \equiv 0$ a.e.

e.g., f with a finite support $\Rightarrow p(f) = 0$.

Consider a quotient space $L/N(p)$ (i.e., equiv. modulo $N(p)$).

$N(p) = \{ f \in L \mid p(f) = 0 \}$.

That is, for $x, y \in L$, we write $[x], [y] \in L/N(p)$, and $[x] = \{ x + n \in L \mid n \in N(p) \}$, $[x] = [y]$ if $x - y \in N(p)$. Define the norm $\| [x] \|_{L/N(p)} := \inf_{y \in N(p)} p(x-y)$

Then $L/N(p)$ becomes a normed space. Let $\mathcal{H}$ be the completion of $L/N(p)$ w.r.t. $\| \cdot \|_{L/N(p)}$ and the inner prod.

$$\langle f, g \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x) \overline{g(x)} \, dx .$$

Now the fcn's $\{ e^{i\lambda t} \}_{\lambda \in \mathbb{R}}$ are pairwise ON elements of $\mathcal{H}$.

$\Rightarrow \exists$ uncountable ON seq. in $\mathcal{H}$. $\Rightarrow \mathcal{H}$: not separable. //

This space is called the space of almost periodic fcns and denoted by B^2 to honor A.S. Besicovitch.

☆ Compactness of the inverse of an RSL operator

Our foregoing discussion already proved:

Thm Suppose 0 is not an eigval of $\mathcal{L}$ (an RSL op.). Let $\mathcal{K}$ be the integral op. on $L^2(a,b)$ with $k(x,\tau)$ as the Green's fcn. Then $\mathcal{K}$ is a compact self-adj. op. and for $\forall g \in C[a,b]$, $\mathcal{L}f = g \Rightarrow f = \mathcal{K}g$. ///

Remark: This thm almost states that $\mathcal{K} = \mathcal{L}^{-1}$. However, two things prevent us from stating so:

① $g \in C[a,b]$, not yet shown for $g \in L^2(a,b)$.

② $\underbrace{\text{Range}(\mathcal{L})}_{= D(\mathcal{L}^{-1})} \subsetneqq L^2(a,b)$ at this point due to the B.C. imposed.

Can we extend $D(\mathcal{L})$ so that $\text{Range}(\mathcal{L}) = L^2(a,b)$?
$\Rightarrow$ Yes! But, properly resolving ② requires measure theory. So, we'll avoid dealing with ②.
Here, we can say a bit more about ①:

Lemma Suppose 0 is not an eigval of $\mathcal{L}$ (an RSL op.) Then $\forall g \in L^2(a,b)$, $\mathcal{K}g$ is differentiable and

(*) $(\mathcal{K}g)' = \varphi u' + \psi v'$ where $\begin{cases} \varphi(x) = \frac{1}{c}\int_x^b v(\tau)g(\tau)\,d\tau, \\ \psi(x) = \frac{1}{c}\int_a^x u(\tau)g(\tau)\,d\tau. \end{cases}$

(Proof) By the def. of $\mathcal{K}$, we know $\mathcal{K}g = \varphi u + \psi v$, $u, v \in C^1[a,b]$ by the Fund. Existence Thm for 2nd order ODEs. If $g \in C[a,b]$, then by the Fund. Thm. Calc. $\mathcal{K}g \in C^1[a,b]$, and $(\mathcal{K}g)' = \varphi' u + \varphi u' + \psi' v + \psi v'$

$= \varphi u' + \psi v'$ $\leftarrow \varphi' u + \psi' v = 0.$

If $g \in L^2(a,b)$ (i.e., could be discontinuous), then $\varphi, \psi \notin C^1[a,b]$ in general.

But (*) does not involve φ', ψ'. So $\exists$ a chance!
Certainly, $\varphi u' + \psi v' \in C[a,b]$.
So, (*) is equivalent to

$$(**)\quad \mathcal{K}g(x) = \mathcal{K}g(a) + \int_a^x (\varphi u' + \psi v')\,d\tau, \quad \forall x \in [a,b]$$

$$\overset{\psi(a)=0}{=} \varphi(a)u(a) + \int_a^x (\varphi u' + \psi v')\,d\tau$$

$$= \frac{1}{c} u(a) \int_a^b v(\tau) g(\tau)\,d\tau + \int_a^x (\varphi u' + \psi v')\,d\tau$$

Let's introduce 3 mappings: $\tilde{\mathcal{K}}, \mathcal{M}, \mathcal{N} : L^2(a,b) \to C[a,b]$.

$$\left.\begin{aligned} \tilde{\mathcal{K}} g &:= \mathcal{K} g \\ \mathcal{M} g(x) &:= \frac{u(a)}{c} \int_a^b v(\tau) g(\tau)\,d\tau \\ \mathcal{N} g(x) &:= \int_a^x (\varphi u' + \psi v')\,d\tau \end{aligned}\right\} \forall g \in L^2(a,b).$$

These are continuous (i.e., bdd.) maps w.r.t. $\|\cdot\|_2$ in $L^2(a,b)$ and $\|\cdot\|_\infty$ in $C[a,b]$.
Let's prove this for $\mathcal{N}$.
An antiderivative of an $L^2(a,b)$ fcn $\in C[a,b]$.
So, $\varphi, \psi \in C[a,b] \Rightarrow \varphi u' + \psi v' \in C[a,b]$.
Now, $\forall x \in [a,b]$.

$$|\mathcal{N} g(x)| \le \int_a^x |\varphi u' + \psi v'|\,d\tau$$

$$\le (b-a)\{\|\varphi\|_\infty \|u'\|_\infty + \|\psi\|_\infty \|v'\|_\infty\}$$

$$|\varphi(x)| = \frac{1}{c}\left|\int_x^b v(\tau) g(\tau)\,d\tau\right|$$

Cauchy-Schwarz

$$\le \frac{1}{c}\int_a^b |v(\tau) g(\tau)|\,d\tau \le \frac{1}{c}\|v\|_2 \|g\|_2$$

Take $\sup_{x\in[a,b]}$ on the LHS. $\Rightarrow \|\varphi\|_\infty \le \frac{1}{c}\|v\|_2 \|g\|_2$

Similarly, $\|\psi\|_\infty \le \frac{1}{c}\|u\|_2 \|g\|_2$

So, $|\mathcal{N} g(x)| \le \frac{b-a}{c}\{\|v\|_2 \|u'\|_\infty + \|u\|_2 \|v'\|_\infty\} \|g\|_2$.

So, $N: L^2(a,b) \longrightarrow C[a,b]$ is a bdd. linear op.
Hence $\tilde{K} - M - N$ is a continuous op. from $L^2(a,b)$ to $C[a,b]$, and $\tilde{K} - M - N \equiv 0$ on $\underbrace{C[a,b]}_{\text{dense in } L^2(a,b)} \subset L^2(a,b)$. So, $\tilde{K} - M - N \equiv 0$ on $L^2(a,b)$. $\Rightarrow$ (**), hence (*), are valid for $\forall g \in L^2(a,b)$. ///

Thm Suppose 0 is not an eigval of $\mathcal{L}$.
Then (i) 0 is not an eigval of K either.
(ii) λ is an eigval of $\mathcal{L} \iff \frac{1}{\lambda}$ is an eigval of K
"$\lambda \in \sigma(\mathcal{L})$" "$\frac{1}{\lambda} \in \sigma(K)$"

Furthermore, the eigfcn of $\mathcal{L}$ corresp. to λ coincides with " K " $\frac{1}{\lambda}$.

(Proof) (i) Suppose $Kg = 0$. Then $\varphi u + \psi v = 0$.
By the lemma on page 3, $\varphi u' + \psi v' = (Kg)' = 0$.

$$\Rightarrow \begin{bmatrix} u & v \\ u' & v' \end{bmatrix} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

By the other lemma ($uv' - vu' \neq 0 \ \forall x \in [a,b]$) we must have $\varphi \equiv 0 \equiv \psi$.

$$\iff \int_x^b v(\tau) g(\tau)\, d\tau \equiv 0 \equiv \int_a^x u(\tau) g(\tau)\, d\tau, \ \forall x \in [a,b].$$

$\iff vg \equiv 0 \equiv ug$ a.e., i.e., in $L^2(a,b)$ sense.
From the above lemma again, u, v cannot vanish simultaneously. $\Rightarrow g \equiv 0$ a.e., i.e., $0 \notin \sigma(K)$. //

(ii) Suppose that $\lambda \in \sigma(\mathcal{L})$ with corresp. eigfcn $f \in D(\mathcal{L})$, i.e., $\mathcal{L}f = \lambda f$, $f \not\equiv 0$, $\lambda \neq 0$. Since $f \in D(\mathcal{L})$, $\lambda f \in C[a,b]$.
So, by the Green's fcn thm, $\mathcal{L}f = \lambda f \iff f = K(\lambda f)$
$= \lambda K f$

Since $\lambda \neq 0$, this is equivalent to $\mathcal{K}f = \frac{1}{\lambda} f$,
i.e., $\frac{1}{\lambda} \in \sigma(\mathcal{K})$ and f is the corresp. eigfcn.
Conversely, suppose that $\mu \in \sigma(\mathcal{K})$ with corresp. eigfcn $g \in L^2(a,b)$
i.e., $\mathcal{K} g = \mu g$, $g \not\equiv 0$, $\mu \neq 0 \iff g = \frac{1}{\mu} \mathcal{K} g \underset{\text{Lemma}}{\Rightarrow} g \in C[a,b]$

Hence, by the Green's fcn thm, $\mathcal{L} f = g$ has the sol.
$f = \mathcal{K} g$. So, $\mathcal{L}\mathcal{K} g = g \iff \mathcal{L} \mu g = g \iff \mathcal{L} g = \frac{1}{\mu} g$. ///