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CAN ONE HEAR THE SHAPE OF A DRUM? REVISITED*

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Abstract. In a landmark paper, Mark Kac in 1966 [Amer. Math. Monthly, 73, pp. 1–23] showed that geometric properties of regions in R^2 can be obtained by studying the asymptotic properties of the spectrum of the Laplacian. He also conjectured that the shape of a region might be completely determined by the spectrum. We describe recent developments in this field and discuss the status of the Kac conjecture as well as similar conjectures for operators other than the Laplacian.

Key words. eigenvalue, universal inequalities, lower bounds

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1. Introduction. The question "Can one hear the shape of a drum?" is silly. Everyone knows that we hear sounds not shapes. Nevertheless this very question is the title of a famous article by Mark Kac which appeared in 1966 [9]. Not only that, the provocative question has spawned articles on the same subject with similar titles. "On hearing the shape of a drum: further results," by Stewartson and Waechter [27] and "Hearing the shape of an annular drum" by Gottlieb [4] are two examples. Moreover, other senses are becoming involved. There is the article by Pinsky [18] titled "Can you feel the shape of a manifold with Brownian motion?"

Actually, the question we are revisiting has two meanings: a mathematical one and a nonmathematical one. We describe both. As we are all aware, the sounds a drum makes when it is struck are determined by its physical characteristics, i.e., the material used, its tautness, and the size and shape. Drums vibrate at certain distinct frequencies called normal modes. The lowest or base frequency is the fundamental tone and the higher frequencies are called overtones.

The nonmathematical interpretation of the question is the following: suppose a drum is being played in one room and a person with perfect pitch, i.e., one who can identify exactly *all* the normal modes of vibration, hears but cannot see the drum. Is it possible for her to deduce the precise shape of the drum just from hearing the fundamental tone and all the overtones?

In his 1966 paper, Kac posed this problem and, although he was unable to provide an answer, he showed that it is possible to get information about the geometry of the drum from a study of its normal modes of vibration. Actually, quite a bit about the geometry had already been found, based principally on the early work of Weyl [31], [32] and extensions due to Pleijel [19], [20], [21]. To glean this information about the shape of a drum on the basis of its normal modes of vibration, we require the equivalent mathematical meaning of the question in the title of Kac's article.

The vibration of a two-dimensional drum which spans a domain Ω in R^2 is governed by the wave equation

$$\frac{\partial^2 v}{\partial t^2}=c^2\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{1}$$

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where $v=v(x,y,t)$ denotes the transverse displacement of a point (x,y) of the drum at time t. For convenience we assume the material, its tautness, and the units of measurement are such that the constant $c^2=1$. It is customary to separate variables in (1) by setting $v=F(t)u(x,y)$, in which case u is a solution of the stationary equation

$$\Delta u+\lambda u=0 \quad \text{in } \Omega. \tag{2}$$

Since the boundary of the drum is firmly attached, solutions of (2) satisfy the boundary condition

$$u=0 \quad \text{on } \partial\Omega. \tag{3}$$

It is classical that there is a countable sequence of eigenvalues

$$0<\lambda_1<\lambda_2\leqq\lambda_3\leqq \cdots \leqq\lambda_k\leqq \cdots \tag{4}$$

tending to $+\infty$ and a sequence of corresponding eigenfunctions $u_1,u_2,\cdots,u_k,\cdots$ such that each u_k satisfies (3) and the equation

$$\Delta u_k+\lambda_k u_k=0 \quad \text{in } \Omega.$$

The eigenfunctions are orthogonal in $L_2(\Omega)$ and they are customarily normalized so that $\|u_k\|_{L_2(\Omega)}=1$ for all k. Two domains, Ω_1 and Ω_2 in R^2, which have the same set of eigenvalues (4) are called *isospectral* and two domains which are congruent in the sense of Euclidean geometry are termed *isometric*. The precise problem raised by Kac is the following: If two domains in R^2 are isospectral, is it necessarily true that they are isometric? This question is the mathematical version of hearing the shape of a drum. Kac tells us that he learned of the mathematical formulation of the problem from S. Bochner who had originally proposed it in the mid 1950's. Then in a casual conversation Kac described the Bochner problem to L. Bers who immediately responded with the intriguing question which identifies the nonmathematical interpretation of the vibrating drum question.

We designate as Problem I the Bochner–Kac question and, in order to describe fully the developments since 1966, we pose three additional related problems. In Problem II we consider again a vibrating drum but with a boundary condition different from (2). The equation

$$\Delta u+\lambda u=0 \quad \text{in } \Omega$$

is the same as in Problem I, but here we require that the normal derivative of u vanish on the boundary, i.e.,

$$\frac{\partial u}{\partial n}=0 \quad \text{on } \partial\Omega, \tag{4a}$$

where $\partial/\partial n$ is the directional derivative normal to the boundary of Ω at each point. Problem II corresponds to the motion of a drum in which the drum material rests on the rim of the drum rather than being fastened. Here also there is a sequence of eigenvalues $0=\mu_1<\mu_2\leqq\mu_3\leqq \cdots \leqq\mu_k\leqq \cdots$ corresponding to the normal modes of vibration as in Problem I. To each eigenvalue there is an eigenfunction and these form a complete orthogonal system in Ω. Problem II asks whether or not two nonisometric domains Ω_1 and Ω_2 can be isospectral with respect to solutions (2), (4a).

Problems I and II can be treated as special cases of the problem

$$\Delta u+\tau u=0 \quad \text{in } \Omega$$

with the boundary condition

$$\alpha u+\beta\frac{\partial u}{\partial n}=0 \quad \text{on } \partial\Omega.$$

Pleijel's methods were applied to this problem for convex domains in R^2 by Sleeman and Zayed [25].

Problems similar to I and II can be posed for general second order elliptic operators, for higher order elliptic operators, and for elliptic systems. For simplicity we confine ourselves to two additional typical problems: one for a higher order operator and one for a specific physical system.

The vibration of a stiff plate differs from that of a membrane not only in the equation which governs its motion but also in the way the plate is fastened to its boundary. A plate spanning a domain Ω in R^2 has its transverse vibrations governed by the (stationary) equation

$$(5) \qquad \Delta^2 u-\nu u=0 \quad \text{in } \Omega$$

with the boundary conditions

$$(6) \qquad u=0 \quad \text{and} \quad \frac{\partial u}{\partial n}=0 \quad \text{on } \partial\Omega.$$

That is, not only is the rim of the plate firmly fastened to the boundary, but the plate is clamped so that no lateral motion can occur at the edge. The problem (5), (6), designated III, also has a set of normal modes of vibration $0<\nu_1\leqq\nu_2\leqq \cdots \leqq\nu_k\leqq \cdots$ and a corresponding set of eigenfunctions. We again ask the question whether or not two clamped plates of different shapes can be isospectral.

Let Ω be a domain in R^3 and consider the eigenvalue problem for the equations of classical elasticity. With $\mathbf{u}=(u_1,u_2,u_3)$ the elastic displacement vector, a function from Ω into R^3, the stationary system governing the modes of elastic vibration is

$$(7) \qquad \mu\Delta u_p+(\lambda+\mu)\frac{\partial}{\partial x_p}(\nabla\cdot\mathbf{u})+\sigma u_p=0 \quad \text{in } \Omega, \qquad p=1,2,3,$$

and typically $\mathbf{u}$ satisfies the boundary conditions

$$(8) \qquad u_p=0 \quad \text{on } \partial\Omega, \qquad p=1,2,3.$$

Here λ and μ are the Lamé constants. The system (7), (8) has a sequence of eigenfunction solutions corresponding to the eigenvalues $0<\sigma_1\leqq\sigma_2\leqq \cdots \leqq\sigma_k\leqq \cdots$. Problem IV raises the same question for (7), (8) with respect to two domains Ω_1 and Ω_2 in R^3 that Problems I-III raise for domains in R^2.

Even before the appearance of Kac's paper, Milnor considered problems analogous to Problem I for compact manifolds. That is, suppose M_1 and M_2 are compact manifolds in R^n, $n\geqq 2$. It is known that for each such manifold M there is a sequence of eigenvalues $0=\lambda_1\leqq\lambda_2\leqq \cdots \leqq\lambda_k\leqq \cdots$ and eigenfunctions $u_1,u_2,\cdots,u_k,\cdots$ which solve

$$\Delta u_k+\lambda_k u_k=0 \quad \text{in } M.$$

We note that since M has no boundary, the eigenfunctions u need not satisfy any boundary condition. In 1964 Milnor showed [15] that there exist two 16-dimensional tori which have the same spectrum but which are not isometric. Thus for compact manifolds the question raised by Kac had already been answered in the negative—at least for manifolds in 16-dimensional space. Later Kneser [10] exhibited two 12-dimensional tori which are isospectral but which are not isometric. In 1980 Vignéras [30] solved the compact manifold problem in all dimensions $n \geqq 2$ by exhibiting for each n two isospectral manifolds which are not isometric. On the other hand, she remarks that in two dimensions, tori which are isospectral must be isometric.[1] Thus for dimensions n between 3 and 11, it is still not known if there exist two isospectral tori which are not isometric.

In each of Problems I through IV, the eigenvalues form a discrete sequence which tends to infinity. In §2 we study the asymptotic behavior of these sequences, and the relationships of such behavior to the geometry of the domain Ω. In §3 we discuss inequalities among the eigenvalues themselves. It turns out that in all dimensions certain inequalities hold among the eigenvalues regardless of the size or shape of the domain. We call these *universal inequalities*, and we shall see that among all sequences of positive numbers tending to infinity those which correspond to sequences of eigenvalues are highly restricted. In §4 we describe bounds for the first eigenvalue, i.e., the fundamental tone in Problem I. Here too we find rather sharp restrictions on the first eigenvalue, ones which yield information on the geometry of the domain. Finally, in §5 we show how recent results give a partial response to the original question raised by Kac. It turns out that for Problems I and II there are domains in R^n, $n \geq 4$, which are isospectral but not isometric. The problem remains unsolved for $n=2,3$ and in all dimensions for Problems III and IV.

2. The asymptotic behavior of eigenvalues. It is known that the sequence of eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k, \cdots$ for Problem I now considered for a domain Ω contained in R^n, $n \geqq 2$, has the property that $\lambda_k \to \infty$ as $k \to \infty$. In the early part of this century, Hilbert conjectured that a study of the asymptotic behavior of this sequence, while extremely difficult, would yield results of the utmost importance. In 1911 Weyl [31], in a brilliant solution to the problem, showed that

$$(1) \qquad \lambda_k \sim 4\pi^2 \left(\frac{k}{B_n V} \right)^{2/n}, \qquad k \to \infty$$

where V is the volume of Ω and B_n is the volume of the unit ball in R^n. The same formula holds for Problem II, i.e.,

$$(2) \qquad \mu_k \sim 4\pi^2 \left(\frac{k}{B_n V} \right)^{2/n}, \qquad k \to \infty.$$

We see that formulas (1) and (2) immediately shed light on the problem of Bochner–Kac. Two domains with different volumes can never have the same spectrum. With the Weyl formula as a startng point, Pleijel [21] in 1954 obtained an additional term in the asymptotic expansion. For any domain Ω in R^2 with area V and length of boundary L, he showed that

$$(3) \qquad \sum_{k=1}^{\infty} e^{-\lambda_k t} \sim \frac{V}{2\pi t} - \frac{L}{4} \frac{1}{\sqrt{2\pi t}} \quad \text{as } t \to 0.$$

[1] M. Berger, P. Gauduchon and E. Mazet, Le spectre d'une variété Riemanniènne, Springer-Verlag, New York–Heidelberg–Berlin, 1971, p. 149.

By a Tauberian theorem the asymptotic formula for the first term on the right side of (3) is equivalent to the Weyl formula (1). However, the second term of (3) when combined with the isoperimetric inequality between the area V and the length of the boundary L for regions in R^2, i.e., $V \leqq \pi L^2/4$, with equality only for the disk, shows that there can be no region in the plane which is isospectral with the disk. Actually, Pleijel showed more. For simply connected domains he established the formula

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} \sim \frac{V}{2\pi t} + \frac{L}{4}\frac{1}{\sqrt{2\pi t}} + \frac{1}{6} \quad \text{as } t \to 0, \tag{4}$$

and in fact he showed that an additional term may be added to the right side of (4), one involving the curvature of the boundary of Ω. Kac uses a combination of probability techniques and heat equation methods to establish (3) for convex domains, and he obtains (4) as a limiting case of convex polygonal domains. He also conjectures that for multiply connected domains in R^2 with r holes, the number $\frac{1}{6}$ in (4) should be replaced by $\frac{1}{6}(1-r)$. McKean and Singer [13] in an extension of the results of Pleijel answered affirmatively the conjecture of Kac with respect to the third term for multiply connected domains. For regions in R^n, McKean and Singer also obtain information about the curvature of the boundary of Ω. They obtain the asymptotic formula for Problem I in the case of general second order elliptic operators. Formula (4) is the first three terms of a general asymptotic expansion (in R^2) of the form

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} \sim a_{-1}t^{-1} + a_{-1/2}t^{-1/2} + a_0 + a_{1/2}t^{1/2} + \cdots + a_s t^s + \cdots, \qquad t \downarrow 0$$

in which the powers of t increase in increments of $\frac{1}{2}$. Stewartson and Waechter [27] found the values of a_s for the first five terms and L. Smith [26] found the sixth term in the expansion. The higher coefficients all involve integrals of the curvature of the boundary and its derivatives. Recently Melrose [14] made a careful analysis of these coefficients and showed that in a certain topology bounded C^∞ domains which are isospectral to a given domain form a compact set.

Polya established an important set of inequalities for Problems I and II, ones which show that in many cases the Weyl asymptotic limit is always approached from one side. A domain in R^n is called a *tiling domain* if copies of it fill all of R^n without overlapping or leaving any gaps. Parallelograms and regular hexagons are simple examples of tiling domains in R^2. However, triangles, domains in the shape of a cross and many others easily come to mind as examples of tiling domains. In 1961 Polya [22] showed that if Ω is any tiling domain in R^n, then for *all* integers k,

$$\lambda_k \geqq 4\pi^2 \left(\frac{k}{B_n V} \right)^{2/n}. \tag{5}$$

In other words, not only is the Weyl asymptotic limit approached from above, but the inequality holds even for small values of k. Furthermore, with only a small additional technical restriction on the type of tiling domain, Polya showed that the eigenvalues for Problem II satisfy the inequality

$$\mu_k \leqq 4\pi^2 \left(\frac{k-1}{B_n V} \right)^{2/n},$$

and this bound holds for $k = 1, 2, \cdots$. That is, the Weyl asymptotic limit in this case is approached from below.

It is well known that the eigenvalues $\nu_1, \nu_2, \cdots, \nu_k, \cdots$ for the clamped plate (Problem III) are related to the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k, \cdots$ for the membrane (Problem I) by the inequalities

$$\nu_k \geqq \lambda_k^2, \qquad k=1,2,\cdots. \tag{6}$$

Polya's result (as he points out) yields for tiling domains the inequalities

$$\nu_k \geqq 16\pi^4 \left(\frac{k}{B_n V}\right)^{4/n}, \qquad k=1,4,\cdots.$$

Now Pleijel [20] in an extension of Weyl's formula to higher order equations established the formula

$$\nu_k \sim 16\pi^4 \left(\frac{k}{B_n V}\right)^{4/n} \quad \text{as } k \to \infty.$$

We see that Polya's result states that the eigenvalues for the clamped plate tend to their limit from above, at least for tiling domains.

Polya conjectured that the results he obtained for tiling domains are valid for arbitrary bounded domains in R^n. As for Problem II there has been no progress thus far on this conjecture. However, Li and Yau [12] attacked the Polya conjecture for Problem I and obtained a somewhat weaker set of inequalities, but for arbitrary bounded domains. They showed that for all k, and for any bounded domain Ω in R^n,

$$\lambda_k \geqq \frac{n}{n+2} 4\pi^2 \left(\frac{k}{B_n V}\right)^{2/n}. \tag{7}$$

In fact, they established a result somewhat stronger than (7) by showing that

$$\sum_{i=1}^{k} \lambda_k \geqq \frac{nk}{n+2} 4\pi^2 \left(\frac{k}{B_n V}\right)^{2/n}. \tag{8}$$

Then (7) is a direct consequence of (8) which we get by replacing all the λ_i in the left side of (8) by λ_n. With the aid of (7) it is possible to obtain a one-sided set of inequalities for Problem III. Using (6) we see at once that for arbitrary bounded domains in R^n,

$$\nu_k \geqq \left(\frac{n}{n+2}\right)^2 16\pi^4 \left(\frac{k}{B_n V}\right)^{4/n}. \tag{9}$$

The results of Li and Yau have been extended by Levine and Protter [11] to certain higher order equations and to classes of elliptic systems. The inequalities we obtain for Problem III are

$$\sum_{j=1}^{k} \nu_j \geqq \frac{nk}{n+4} 16\pi^4 \left(\frac{k}{B_n V}\right)^{4/n}. \tag{10}$$

In particular, if all the ν_j are replaced by ν_k in (10), we get the weaker inequality

$$\nu_k \geqq \frac{n}{n+4} 16\pi^4 \left(\frac{k}{B_n V}\right)^{4/n} \tag{11}$$

which is stronger than (9), the one implied by Li and Yau. However, it is still weaker than the analogue of the Polya conjecture. In that case we would replace $n/(n+4)$ by 1 in (11).

For the elliptic system describing the elastic vibration of a domain in R^3, i.e., Problem IV, Pleijel [19] obtained an asymptotic formula for the eigenvalues which takes the form

$$\sigma_k \sim \mu \left(\frac{6\pi^2}{\left[2+(2+\lambda/\mu)^{-3/2}\right]} \right)^{2/3} \left(\frac{k}{V} \right)^{2/3} \quad \text{as } k \to \infty. \tag{12}$$

It is natural to extend the Polya conjecture to this system and expect that for all k,

$$\sigma_k \geqq \mu \left(\frac{6\pi^2}{\left[2+(2+\lambda/\mu)^{-3/2}\right]} \right)^{2/3} \left(\frac{k}{V} \right)^{2/3}. \tag{13}$$

Levine and Protter [11] show that

$$\sum_{j=1}^{k} \sigma_j \geqq \frac{3\mu k}{5} \left(\frac{2\pi^2 k}{V} \right)^{2/3}$$

for all k, from which the inequality

$$\sigma_k \geqq \frac{3\mu}{5} \left(\frac{2\pi^2 k}{V} \right)^{2/3} \tag{14}$$

follows. While (14) is much weaker than (13), it is the first such one-sided lower bound known which is valid for all the eigenvalues of the elasticity system.

3. Universal inequalities. It is clear that the problem of Kac is not going to be solved merely by a consideration of the asymptotic behavior of the eigenvalues, as discussed in §2. Two sets of eigenvalues may be identical except for the first few, and thus an examination of the asymptotics would in no way distinguish between the sets. Of course, it may turn out that if two sets of eigenvalues $\lambda'_1, \lambda'_2, \cdots, \lambda'_k, \cdots$ and $\lambda''_1, \lambda''_2, \cdots, \lambda''_k, \cdots$ are identical for all sufficiently large k, then the sets themselves must be the same. If that turns out to be correct, then it may be possible to identify domains solely by the asymptotic behavior of the eigenvalues of the Laplacian.

Let $\mathscr{S}$ be the set of all increasing sequences of nonnegative numbers which tend to infinity and let A be a given elliptic operator. An inverse problem, one which is probably more important than the problem considered by Kac, was raised by Stewartson and Waechter [27] although they only mention the problem in passing. The question we raise is that of identifying those sequences in $\mathscr{S}$ which correspond to spectra of A for some domain. That the subset of $\mathscr{S}$ is severely restricted is clear from the formulas of Weyl, Pleijel, and others. In this section we establish a class of universal inequalities which sharply restrict further those sequences which are spectral.

The initial result in this direction is due to Payne, Polya, and Weinberger [17]. They show that for any region in R^2, the eigenvalues for Problem I satisfy the inequality

$$\frac{1}{\lambda_{k+1}-\lambda_k} \sum_{i=1}^{k} \lambda_i \geqq \tfrac{1}{2} k \quad \text{for } k=1,2,\cdots. \tag{1}$$

In fact, the simple (and weaker) inequality

$$\lambda_{k+1} \leqq 3\lambda_k \tag{2}$$

is obtained by replacing all the λ_i by λ_k in the left side of (1). One of the most interesting features of (1) and (2) is that the inequalities are *independent of the size or shape of the domain*. Also, the result is valid for all k. Thus no sequence in $\mathscr{S}$ with the property, say, that one number in the sequence is more than three times its immediate predecessor, can be in the spectrum of a domain in R^2.

The result (1) has an immediate extension to domains in R^n. The appropriate inequality is

$$\frac{1}{\lambda_{k+1}-\lambda_k}\sum_{i=1}^{k}\lambda_i \geqq \frac{nk}{4}, \qquad k=1,2,\cdots, \tag{3}$$

which yields the result

$$\lambda_{k+1} \leqq \frac{n+4}{n}\lambda_k. \tag{4}$$

We note that as the dimension increases, the eigenvalues get closer together. In terms of vibrating drums we may interpret this statement by saying that the overtones for high dimensional drums are more difficult to distinguish than those for low dimensional drums.

Inequality (3) has been sharpened and improved by Hile and Protter [7]. For any domain in R^n, the eigenvalues of Problem I satisfy the inequality

$$\sum_{i=1}^{k}\frac{\lambda_i}{\lambda_{k+1}-\lambda_i} \geqq \frac{nk}{4}, \qquad k=1,2,\cdots. \tag{5}$$

We note that (5) reduces to (3) if all the λ_i in the denominator of the left side of (5) are replaced by λ_k. Hence (3) is weaker than (5) for all $k \geqq 2$. Corresponding to (4), the inequality (5) yields the sequence of inequalities

$$\lambda_{k+1} \leqq c(n,k)\lambda_k,$$

where the $c(n,k)$ are obtained from (5) by solving a polynomial equation of degree k. For all $k \geqq 2$, the number $c(n,k)$ is strictly less than $(n+4)/n$.

Payne, Polya, and Weinberger established a set of universal inequalities for Problem III [17]. The result in this case is

$$\frac{1}{\nu_{k+1}-\nu_k}\sum_{i=1}^{k}\nu_i \geqq \frac{k}{8} \tag{6}$$

for domains in R^2. Using the technique developed in [7], Hile and Yeh [8] extended and improved (6) to obtain

$$\sum_{i=1}^{k}\frac{\sqrt{\nu_i}}{\nu_{k+1}-\nu_i} \geqq \frac{n^2k^{3/2}}{8(n+2)}\left(\sum_{i=1}^{k}\nu_i\right)^{-1/2} \tag{7}$$

for domains in R^n. Inequality (7) is stronger than (6) for all $k \geqq 2$.

It is interesting that no universal inequalities such as (3), (5) or (6), (7) have been obtained for Problem II. The technique used in [7], [8], and [17] just does not seem to work, although it is most likely that some type of universal inequalities exist for the

Neumann problem (perhaps involving both λ_i and μ_i) for the Laplacian, i.e., for Problem II. The situation for systems as exemplified by Problem IV is even murkier, and it may well be the case that universal inequalities for the eigenvalues of elliptic systems are not to be found.

The results of Hile and Yeh have recently been extended by Zu-chi Chen [3] to yield universal inequalities for the eigenvalues of a subclass of equations of the form

$$\Delta^{2m}u - \Gamma u = 0$$

with the boundary conditions

$$u = \frac{\partial u}{\partial n} = \cdots = \frac{\partial^{m-1}u}{\partial n^{m-1}} = 0.$$

The result is

$$\sum_{i=1}^{k} \frac{\Gamma_i^{1/2m}}{\Gamma_{k+1} - \Gamma_i} \geq \frac{n^2 k^{(4m-1)/2m}}{8m(4m+n-2)} \left(\sum_{i=1}^{k} \Gamma_i \right)^{(1-2m)/2m}, \tag{8}$$

valid for certain values of m. It has recently been shown by S. Hook (unpublished) that (8) holds for all values of m, and that a similar inequality holds for the operator Δ^p where p is any integer greater than 1. In fact, these turn out to be special cases of a somewhat stronger inequality valid for a class of higher order operators of which the powers of the Laplacian are special cases.

4. Bounds for the lowest eigenvalue. The quality of the sound of a drum is influenced most by the fundamental tone. Similarly, the vibration as well as the buckling properties of a clamped plate are determined largely by the first eigenvalue and its corresponding eigenfunction. We can obtain approximate values of the eigenvalues of Problem I most easily by means of the variational definition of the spectrum. Let $\mathscr{F}$ be the class of functions which are continuous in a domain Ω, vanish on $\partial\Omega$, and have piecewise continuous first derivatives. Then λ_1 of Problem I is defined by the solution of the minimization problem

$$\lambda_1 = \inf_{v \in \mathscr{F}} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega v^2 dx}. \tag{1}$$

If $u_1, u_2, \cdots, u_{k-1}$ are the eigenfunctions corresponding to $\lambda_1, \lambda_2, \cdots, \lambda_{k-1}$, then λ_k is defined by

$$\lambda_k = \inf_{v} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega v^2 dx} \tag{2}$$

where the infimum is taken over all functions v in $\mathscr{F}$ which are orthogonal (in $L_2(\Omega)$) to $u_1, u_2, \cdots, u_{k-1}$. We find upper bounds for λ_1 simply by inserting any function of $\mathscr{F}$ in the right side of (1), which is known as the Rayleigh quotient. The well-known Rayleigh–Ritz method for approximating eigenvalues is based on this fact. As is evident from the variational definition of λ_1 as a minimum problem, lower bounds for λ_1 are difficult to find. Let $P = (P_1, P_2, \cdots, P_n)$ be any vector field defined in Ω such that $P_i(x_1, \cdots, x_n)$ is continuous in x and C^1 in the variable x_i. Hersch [6] and Protter [23] at about the same time showed that λ_1 may be estimated from below by the inequality

$$\lambda_1 \geqq \inf_{x \in \Omega} \left(\operatorname{div} P - |P|^2 \right) \tag{3}$$

where $|P|$ denotes the Euclidean norm of P. It is important to note that no boundary conditions are imposed on the function P.

From (1) it is not difficult to see that the first eigenvalue for Problem I satisfies an *inclusion principle*, i.e., if $\Omega_1 \subset \Omega_2$ then λ_1 for Ω_1 must be larger than (or equal to) λ_1 for Ω_2. In fact, this principle can be used to compare all the eigenvalues of the spectrum of Ω_1 with the corresponding ones for Ω_2. Our experience tells us that larger drums have lower fundamental tones than smaller ones. It is natural to ask whether or not the fundamental tone tends to zero as the area of the drum (or the volume if $n > 2$) tends to infinity. The simple case of a rectangular drum shows that the magnitude of λ_1 is not directly related to the area of a drum. If the length of a rectangular drum is πa and its width is πb, then $\lambda_1 = 1/a^2 + 1/b^2$. For sufficiently small values of a and sufficiently large values of b, the area of the drum can be made arbitrarily large while the value of λ_1 also is arbitrarily large. On the other hand, for circular drums, $\lambda_1 \to 0$ as the radius of the drum tends to infinity. Polya conjectured that in order for a drum of any shape to have a low fundamental tone the shape must be such that it contains in its interior a large circular drum. The first step in establishing this conjecture was made by Hersch [6] who used (3) to show that for convex drums Ω in R^2, the inequality

$$\lambda_1 \geqq \frac{\pi^2}{4\rho^2} \tag{4}$$

holds, where ρ is the radius of the largest inscribed disk in Ω. Inequality (4) is of great interest since it shows that no matter how large the area of a domain may be, the fundamental tone will be very high if the shape is such that only a very small disk can be fitted into the region.

It is not difficult to show that (4) also holds for domains in R^n, $n \geqq 3$, with ρ the radius of the largest inscribed n-dimensional ball. The inequality (4) is sharp in R^2 since equality holds for an infinite strip. For compact convex domains in R^n, a stronger inequality can be established. Let d be the supremum of the distance between any two points of Ω. Then the inequality

$$\lambda_1 \geqq \frac{\pi^2}{4} \left(\frac{1}{\rho^2} + \frac{n-1}{d^2} \right)$$

holds [24], a clear improvement of (4) for any bounded convex domain.

It turns out that the Polya conjecture is valid for general domains. By an ingenious method, Hayman [5] showed that for any domain in R^2 with a reasonable boundary,

$$\lambda_1 \geqq \frac{1}{900\rho^2}.$$

He also obtained an inequality of the form

$$\lambda_1 \geqq \frac{c_1}{\rho^2}$$

for domains in R^n, $n>2$, but in the higher dimensional case, he required certain restrictions on the boundary of the domain. Moreover, the constant c_1 has not been evaluated. Since Hayman's breakthrough, improvements in the constant in R^2 have been made. Osserman [16] showed that for $n=2$,

$$\lambda_1 \geqq \frac{1}{4\rho^2}.$$

Also, estimates have been obtained for multiply-connected domains of finite connectivity. Both Osserman and Taylor [28] have lower bounds for λ_1 in terms of ρ and the connectivity of the domain.

5. Isospectral domains. It is natural to ask whether the Polya inequality

$$(1) \qquad \lambda_k \geqq 4\pi^2 \left(\frac{k}{B_n V} \right)^{2/n}, \qquad k=1,2,\cdots$$

is valid for any domains other than tiling domains. Since there are very few shapes for which the eigenvalues of any of the Problems I-IV are known explicitly, the verification or contradiction of (1) is a formidable problem. However, in 1980, Bérard [1] found a large class of nontiling domains for which he was able to calculate explicitly all the eigenvalues for Problems I and II. He discovered, indeed, that in all cases (1) holds. Then Bérard and Besson [2] found additional classes of domains for which they were able to verify (1). Using the work of Bérard and Besson, Urakawa [29] exhibited bounded domains in R^n, $n \geqq 4$, which are isospectral with respect to *both* Problems I and II and which are not isometric. We see that the question originally posed by Bochner has a negative answer in all dimensions greater than three.

To describe the domains found by Bérard, Besson, and Urakawa we first introduce a root system in R^n and define the Weyl reflection group. Let $\alpha_1, \alpha_2, \cdots, \alpha_p$ be a set S of vectors which spans R^n and does not contain the zero vector. Let $H_1, H_2, \cdots, H_p$ be a set H of hyperplanes, each passing through the origin and such that H_i is orthogonal to α_i, $i=1,2,\cdots,p$. We introduce transformations $T_1, T_2, \cdots, T_p$ in R^n such that T_i reflects each point in R^n in the hyperplane H_i. Thus T_i leaves H_i fixed. We assume that the set $\{\alpha_i\}$ has the following two properties:

(i) $2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i) \in \mathbf{Z}$, $i,j=1,2,\cdots,p$, where (α, β) is the Euclidean inner product of the vectors α, β, and

(ii) each transformation T_i leaves the set H invariant.

Any set S satisfying (i) and (ii) is called a *root system.*

The simplest example of a root system is the set of coordinate unit vectors in R^n and their negatives. Conditions (i) and (ii) are severe restrictions on root systems and for each n there are only finitely many. For example, in R^2 there are only four possible root systems. One such consists of the eight vectors $\pm i$, $\pm j$, $\pm(i+j)$, $\pm(i-j)$ where i and j are unit coordinate vectors; another consists of the vectors $\pm i$, $\pm(i \pm \sqrt{3} j)$. The only other root system in R^2 (except for the coordinate vectors) contains 12 vectors.

The hyperplanes $H_1, \cdots, H_p$ divide R^n into disjoint polyhedral cones, each with a vertex at the origin. These are called *Weyl chambers*. The hyperplanes $\{H_i\}$ may be written in coordinates as solutions of the equation $\alpha_i \cdot x = 0$ for $x \in R^n$. Thus the hyperplanes $\alpha_i \cdot x = k$ with $k \in \mathbf{Z}$ form a system of hyperplanes parallel to those in H. The totality of such planes divides R^n into a set of bounded polyhedral regions. Among this set we distinguish those which have the origin for one of its vertices. Such polyhedrons are called *alcoves*.

Bérard and Besson [2] found explicit expressions for the eigenvalues of Problems I and II for alcoves, and they verified that the Polya inequality for tiling domains is also valid for alcoves. They also found explicit expressions for all the eigenvalues of those domains in S^{n-1} (the unit sphere in R^n) cut off by Weyl chambers. Here again the Polya inequality was verified.

Urakawa [29] considered the following type of region. Let $B(0,\varepsilon)$ and $B(0,1)$ be balls in R^n with center at the origin and radius $\varepsilon<1$ and 1, respectively. Denote by U the portion of a Weyl chamber contained in the annulus $B(0,1)/B(0,\varepsilon)$. Then using the results of Bérard and Besson, Urakawa found for $n\geq 4$ two such regions, U and U', which are isospectral with respect to Problems I and II but which are not isometric. Urakawa accomplished this task by finding explicit expressions for the eigenvalues of U and U' and then using properties of Lie algebras to show that the two domains are not isometric. As Urakawa points out, not only is the problem for $n=2$ and 3 still open, but the fact that domains such as U and U' have sharp edges leaves unsolved the question of whether or not smooth isospectral domains are always isometric. This is not merely a technical point, as in dimension 2 Kac shows that domains with corners have asymptotic eigenvalue expansions which differ sharply from those without corners. The appearance of cusps introduces other changes as Stewartson and Waechter [27] show.

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