LOCALIZATION OF FRAMES, BANACH FRAMES, AND THE INVERTIBILITY OF THE FRAME OPERATOR

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ABSTRACT. We introduce a new concept to describe the localization of frames. In our main result we shown that the frame operator preserves this localization and that the dual frame possesses the same localization property. As an application we show that certain frames for Hilbert spaces extend automatically to Banach frames. Using this abstract theory, we derive new results on the construction of nonuniform Gabor frames and solve a problem about non-uniform sampling in shift-invariant spaces.

1. INTRODUCTION

Frames are a tool for the construction of series expansions in Hilbert spaces. Frames provide stable expansions, but they may be overcomplete and the coefficients in the frame expansion therefore need not be unique, quite in contrast to orthogonal expansions. This redundancy may also be used to advantage, for example, in applications to noise reduction or for the reconstruction from lossy data [18, 22, 40]. In addition, the construction of frames is easier and more flexible than the construction of orthonormal bases.

Although the concept of frames is associated with Hilbert spaces, frames are often used to analyze additional properties of functions besides their membership in that Hilbert space. Their usefulness in applications stems from the fact that subtler properties are often encoded in the frame coefficients. For example, wavelet frames encode information on the smoothness and singularity properties of distributions; Gabor frames (Weyl-Heisenberg frames) encode time-frequency information; and frames consisting of reproducing kernels in certain Hilbert spaces encode pointwise information and yield sampling theorems.

In all these applications, the goal is to recognize the finer properties of functions by means of the magnitudes of the frame coefficients. These properties, typically smoothness and decay properties or phase-space localization of functions, are measured by Banach space norms. In other words, the emphasis lies on the characterization of an associated family of Banach spaces of functions by the values of the frame coefficients. These same Banach spaces also play a crucial role in non-linear approximation and in compression algorithms [21].

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To illustrate this point, we consider a famous example from time-frequency analysis. Take a Gabor frame of the form $\{M_{\beta l}T_{\alpha k}g(t) = e^{2\pi i \beta l \cdot t} g(t-\alpha k) : k, l \in \mathbb{Z}^d\}$ for some window function $g \in L^2(\mathbb{R}^d)$. The frame coefficient $\langle f, M_{\beta l}T_{\alpha k}g\rangle$ then is a measure of the strength of the frequency band at βl at time αk. In quantum mechanics the quantity $|\langle f, M_{\beta l}T_{\alpha k}g\rangle|^2$ is interpreted as the energy of a particle in the state f near the point $(\alpha k, \beta l)$ in phase space. For these interpretations to make sense and to be useful, it is necessary that the window g itself possess good time-frequency concentration.

To see this, we consider two extreme cases. If $g = \chi_{[0,1]^d}$, then $\{M_{\beta l}T_{\alpha k}g : k, l \in \mathbb{Z}^d\}$ is a frame whenever $\alpha \leq 1$ and $\beta \leq 1$ [19, 33]. However, while the frame coefficients $\langle f, M_{\beta l}T_{\alpha k}\chi\rangle$ reveal the localization properties of f in time (because χ has compact support), they do not provide any useful frequency information about f because $\hat{\chi}$ is badly localized. See [33, p. 119] for a discussion. For instance, with this frame it is not even possible to distinguish a Schwartz function from a rapidly decaying step function solely from the magnitudes of the frame coefficients. Therefore this particular frame, although perfectly satisfactory for L^2-theory, cannot and should be not used for purposes of time-frequency analysis.

On the other hand, if the window is the Gaussian $g(t) = e^{-\pi t^2}$, then $\{M_{\beta l}T_{\alpha k}g : k, l \in \mathbb{Z}^d\}$ is a frame for $L^2(\mathbb{R})$ if and only if $\alpha\beta < 1$ [45]. This frame is perfectly suited for the time-frequency analysis of functions and distributions [33, Ch.13]; for example, it can be shown that a function f belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ if and only if the coefficients $\langle f, M_{\beta l}T_{\alpha k}g\rangle$ decay rapidly in $k, l \in \mathbb{Z}^d$ [33, 36]. This frame can be used for purposes of time-frequency analysis.

Current research on frames can be roughly divided into two directions: One objective is the construction and characterization of all frames with a given structure, for example, all Gabor frames or all wavelet frames. See for instance [14, 16, 37, 43, 44]. Another objective is to achieve an understanding of those properties that make a frame useful and the subsequent construction of such frames. In the literature these issues are usually treated separately under the topics "frame theory" and "atomic decompositions".

In this paper we make an attempt to reconcile these diverging aspects of frame analysis. In our opinion it is important to carry frame theory beyond the analysis of Hilbert spaces. In this respect, the combination of ideas from the abstract theory of function spaces with the theory of frames proves to be very fertile and leads to a deeper understanding of exactly which properties of functions can be detected and extracted from the frame coefficients.

We will follow three basic principles.

Principle I. *Associated to each Hilbert space and each Riesz basis there is a natural family of Banach spaces.*

Principle II. *Frames are only useful if they yield a description and characterization of the associated Banach spaces.* In theory, this is accomplished by the concept of *Banach frames* [30]. We therefore try to understand those frames that are also Banach frames for the associated Banach spaces.

These principles are not new, they are implicit in the theory of atomic decompositions of many function spaces. The φ-transform of Frazier and Jawerth is in fact a construction of $L^2(\mathbb{R}^d)$-frames which at the same time yield characterizations of the Besov-Triebel-Lizorkin spaces [28, 29]. Likewise, the Gabor frames of Feichtinger and Gröchenig provide characterizations of modulation spaces and open the door to the time-frequency analysis of distributions [23, 25]. The sampling theory in [6] amounts to the construction of Banach frames consisting of reproducing kernels for a large class of shift-invariant spaces. The atomic decompositions of Coifman and Rochberg [17] and of Janson, Peetre, and Rochberg [39] can be seen in the same light. Some of these contributions actually predate the rediscovery of frames by Daubechies, Grossmann, and Meyer [19] and have been interpreted in the context of frames only a posteriori.

Principle III. *Useful frames possess a localization property. Sufficient localization of a frame is a necessary condition for its extension to a Banach frame for the associated Banach spaces.*

Our main result can then be paraphrased as follows. *Localized frames are universal Banach frames for the associated family of Banach spaces.*

These principles are more a philosophic program than a mathematical theory. To fill them with life, we work on two levels.

On an abstract level we define a concept for the localization of a frame with respect to a Riesz basis (Section 3.3). Our main result states that the dual frame possesses the same localization as the original one. This is the key property for extending a frame for a Hilbert space to a Banach frame for the associated Banach spaces. This insight allows us to reverse the usual order for the construction of atomic decompositions and Banach frames. Instead of constructing a Banach frame for all the associated Banach spaces from scratch as in the works mentioned above, we start with the construction of an "ordinary" frame for a Hilbert space. This task is much easier than the construction of a Banach frame and can be accomplished by Hilbert space techniques. Furthermore, we can draw on the previously established characterizations and constructions of Hilbert frames. In the next step we only need to check whether the frame possesses the localization property. Under this condition, this frame is automatically a Banach frame. We hope that, with this procedure, results from pure frame theory will be useful for future applications. We will follow this pattern in the examples in Sections 4 and 5.

The novelty of our approach lies in an appropriate definition of localized frames and the degree of abstraction. The key tool in the mathematical argument is a theorem of Jaffard on the off-diagonal decay of inverse matrices. This theorem is a simple form of a symbolic calculus in certain Banach algebras of infinite matrices. It is a good substitute for Wiener's lemma in situations without sufficient group structure, and has already been successfully used in the construction of wavelet bases [38].

The degree of abstraction is not a purpose in itself. We will apply the abstract theory of localized frames to concrete examples and obtain new results on Gabor frames and new sampling theorems in shift-invariant spaces.

As our first application we study sampling theorems in shift-invariant spaces and solve a conjecture of Aldroubi and the author mentioned in [6]. We show that an L^2-sampling theorem in a shift-invariant space implies automatically a sampling theorem for weighted L^p-spaces. Moreover, the reconstruction from samples in a weighted ℓ^p-space is performed exactly in the same way as for ℓ^2-samples, namely by the frame reconstruction in terms of the the dual frame. Thus the frame techniques used in sampling theory, when applied correctly, possess an extremely strong additional stability property. This type of stability seems to be very desirable in numerical applications. Finally, we derive a similar statement for sampling from averages in shift-invariant spaces.

As a second application we study Gabor frames, in particular non-uniform Gabor frames. We show that the established characterizations of time-frequency concentration by means of modulation space norms carry over to non-uniform Gabor frames. For uniform Gabor frames, such results have already been obtained by means of fairly deep methods from the theory of Banach algebras [33,35]. So far, a similar time-frequency analysis with non-uniform Gabor frames was considered out of reach, because such frames cannot be analyzed with group theoretic arguments. The main difficulty arises from the fact that the dual frame is no longer determined by a single dual window. The axiomatic theory of localized frames offers a guideline for how to prove such a result in the absence of a group structure: namely, once a non-uniform Gabor frame is given for $L^2(\mathbb{R}^d)$, we only need to check its localization properties. For Gabor frames, the abstract concept of localization coincides with an intuitive notion of localization of the short-time Fourier transform. The abstract theory even yields an extension beyond Gabor frames to frames of time-frequency molecules.

In both examples, we find that the required sampling density is independent of the Banach space. The sampling density for stable sampling in a shift-invariant space is completely determined by the L^2-theory, likewise the density of a non-uniform Gabor frame does not depend on the modulation space, but only on $L^2(\mathbb{R}^d)$. This insight is quite surprising, because some previous theories that were based only on the first two principles required a higher density for weighted spaces [3,17,24,30].

The paper is organized as follows. In Section 2 we collect the main properties of polynomial and sub-exponential weight functions. In Section 3 we present an axiomatic theory of the localization of frames. We define a family of Banach spaces associated to each Hilbert space and Riesz basis (Section 3.1) and investigate the behavior of the frame operator on these spaces (Section 3.2). In Section 3.3 we introduce the localization of a frame. The main theorem on the localization of the dual frame is presented in Section 3.4. In Section 4 we apply the abstract theory to sampling theorems in shift-invariant spaces, thereby solving a conjecture of Aldroubi and Gröchenig. In Section 5 we study non-uniform Gabor frames and show that the result about time-frequency characterization of functions and distributions by means of Gabor frames can be adapted to cover the case of non-uniform Gabor frames.

Further applications (to wavelet frames, multiple windows and generators, sampling theorems in other function spaces) will be investigated in future works.

We assume that the reader is familiar with the basic elements of frame theory and Riesz bases [14, 18, 22]. The background on sampling in shift-invariant spaces and an extensive list of references can be found in [6], for time-frequency analysis we refer to [33]. For reasons of length, we will not make any attempt to be self-contained or to motivate these examples.

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2. Weight Functions

We first summarize some elementary properties of weight functions of polynomial and sub-exponential growth.

Index Sets. In the following $\mathcal{N}$ and $\mathcal{X}$ are countable index sets in some $\mathbb{R}^d$ and we may assume that both $\mathcal{X}$ and $\mathcal{N}$ are *separated*, this means that

$$\inf_{x,y\in\mathcal{X}:x\neq y} |x-y| \geq \delta > 0$$

and likewise for $\mathcal{N}$.

When $\mathcal{X}$ is used to index a family of functions on $\mathbb{R}^d$, the index x in $f_x(t)$ indicates that the essential support of f_x is centered at x.

Polynomial and Sub-Exponential Weight Functions. A weight is a non-negative function on $\mathbb{R}^d$ which we may assume without loss of generality to be continuous. A weight m is called *s-moderate*, if there are constants $C, s \geq 0$ such that

$$m(t+x) \leq C(1+|t|)^s m(x) \qquad \text{for all } t, x \in \mathbb{R}^d\,. \tag{1}$$

The weight function occurring in (1),

$$v_s(t) = (1+|t|)^s \qquad \text{for } t \in \mathbb{R}^d\,, \tag{2}$$

is submultiplicative, i.e., v_s satisfies $v_s(t+x) \leq v_s(t)v_s(x)$.

A weight function m is called sub-exponential if there are constants $C, \gamma > 0$ and $0 \leq \beta < 1$ such that

$$m(t+x) \leq Ce^{\gamma|t|^\beta} m(x) \qquad \text{for all } t, x \in \mathbb{R}^d\,. \tag{3}$$

By setting $x = 0$ in (1) and in (3) we see that a an s-moderate weight m grows at most polynomially, i.e., $m(t) \leq C(1+|t|)^s$, and a sub-exponential weight grows at most like $Ce^{\alpha|t|^\beta}$.

The weighted ℓ^p-space $\ell^p_m(\mathcal{X})$ on the index set $\mathcal{X}$ is defined by the norm

$$\|c\|_{\ell^p_m} = \Big(\sum_{x\in\mathcal{X}} |c_x|^p m(x)^p\Big)^{1/p}\,, \tag{4}$$

with the usual modification for $p = \infty$.

In dealing with s-moderate and sub-exponential weights, we will repeatedly use the following lemmas.

Lemma 2.1. *If $\mathcal{X} \subseteq \mathbb{R}^d$ is separated, then for any $s > d$*

$$\sup_{v\in\mathbb{R}^d} \sum_{x\in\mathcal{X}} \big(1+|x-v|\big)^{-s} = C_s < \infty \,. \tag{5}$$

Proof. Fix $v \in \mathbb{R}^d$. Since $|x-y| \geq \delta$ for $x \neq y \in \mathcal{X}$, each box of side length $\alpha = \frac{\delta}{2\sqrt{d}}$ contains at most one point $x - v$. Therefore we may write $x - v = \alpha n_x + \mu_x$ for unique $n_x \in \mathbb{Z}^d$ and $\mu_x \in [-\alpha, \alpha)^d$, and by the choice of α we have $n_x \neq n_y$ for $x \neq y \in \mathcal{X}$. Since $(1+|x+y|)^{-s} \leq (1+|x|)^s(1+|y|)^{-s}$ for $x, y \in \mathbb{R}^d$ and $s \geq 0$, we find that

$$\begin{aligned}
\sum_{x\in\mathcal{X}} \big(1+|x-v|\big)^{-s} &= \sum_{x\in\mathcal{X}} \big(1+|\alpha n_x + \mu_x|\big)^{-s} \\
&\leq \sum_{x\in\mathcal{X}} \big(1+|\mu_x|\big)^{s} \big(1+|\alpha n_x|\big)^{-s} \\
&\leq C_1\big(1 + \sum_{x\in\mathcal{X}: n_x\neq 0} |\alpha n_x|^{-s}\big) \\
&\leq C_1\big(1 + \alpha^{-s} \sum_{n\in\mathbb{Z}^d\setminus\{0\}} |n|^{-s}\big) < \infty \,.
\end{aligned}$$

Clearly this estimate is independent of v, hence the assertion. ∎

Lemma 2.2. *Assume that $\mathcal{X} \subseteq \mathbb{R}^d$ is separated.*

(a) If $s > d$, then

$$\sum_{x\in\mathcal{X}} \Big(1+|x-n|)^{-s}(1+|x-m|)^{-s}\Big) \leq C(1+|n-m|)^{-s} \qquad m, n \in \mathcal{N} \,. \tag{6}$$

(b) There exists $\alpha' \in (0, \alpha)$ such that

$$\sum_{x\in\mathcal{X}} e^{-\alpha|x-n|}\, e^{-\alpha|x-m|} \leq C e^{-\alpha'|m-n|} \,. \tag{7}$$

Proof. (a) We partition $\mathcal{X}$ into $\mathcal{A}_1 := \{x \in \mathcal{X} : |x-n| \leq |n-m|/2\}$ and $\mathcal{A}_2 = \mathcal{X} \setminus \mathcal{A}_1$. If $x \in \mathcal{A}_1$, then $|x-m| \geq |n-m|/2$ and thus

$$\sum_{x\in\mathcal{A}_1} \Big(1+|x-n|)^{-s}(1+|x-m|)^{-s}\Big) \leq \Big(1+|n-m|/2\Big)^{-s} \sum_{x\in\mathcal{A}_1} \Big(1+|x-n|)^{-s}\Big) \,.$$

If $x \in \mathcal{A}_2$, then $|x-n| \geq |n-m|/2$ and again we obtain

$$\sum_{x\in\mathcal{A}_2} \Big(1+|x-n|)^{-s}(1+|x-m|)^{-s}\Big) \leq \Big(1+|n-m|/2\Big)^{-s} \sum_{x\in\mathcal{A}_2} \Big(1+|x-m|)^{-s}\Big) \,.$$

Here the sums over $\mathcal{A}_1$ and $\mathcal{A}_2$ are finite by Lemma 2.1 and $(1+|n-m|/2)^{-s} \leq 2^s(1+|n-m|)^{-s}$. Thus the assertion follows.

(b) By repeating these arguments for the exponential weight $e^{-\alpha|x|}$, we obtain (7) with $\alpha' = \alpha/2$. (We do not need a sharper estimate for α'.) ∎

The next lemma is a substitute for Young's convolution relation in the absence of a group structure. If $\mathcal{N} = \mathcal{X} = \mathbb{Z}^d$, then the following statement follows from Young's Theorem.

Lemma 2.3. *(a) Let* $A_{xn} = (1+|x-n|)^{-s-d-\epsilon}$ *for some* $\epsilon > 0$ *and* $n \in \mathcal{N}, x \in \mathcal{X}$. *Then the operator* A *defined on finite sequences* $(c_n)_{n\in\mathcal{N}}$ *by matrix multiplication* $(Ac)_x = \sum_{n\in\mathcal{N}} A_{xn}c_n$ *extends to a bounded operator from* $\ell^p_m(\mathcal{N})$ *to* $\ell^p_m(\mathcal{X})$ *for all* $p \in [1,\infty]$ *and all* s*-moderate weights* m.

(b) If we define $A_{xn} = e^{-\alpha|x-n|}$, *then* A *maps* $\ell^p_m(\mathcal{N})$ *to* $\ell^p_m(\mathcal{X})$ *for all* $p \in [1,\infty]$ *and all sub-exponential weights* m.

Proof. (a) Similar to the proof of Schur's test, we show first that A is bounded from $\ell^1_m(\mathcal{N})$ to $\ell^1_m(\mathcal{X})$ and from $\ell^\infty_m(\mathcal{N})$ to $\ell^\infty_m(\mathcal{X})$. The lemma then follows by interpolation (see [11] for the interpolation of weighted L^p-spaces).

Boundedness on ℓ^1.

$$\begin{aligned}
\|Ac\|_{\ell^1_m(\mathcal{X})} &= \sum_{x\in\mathcal{X}} \Big|\sum_{n\in\mathcal{N}} A_{xn}c_n\Big| m(x) \\
&\leq \sum_{x\in\mathcal{X}}\sum_{n\in\mathcal{N}} (1+|x-n|)^{-s-d-\epsilon}|c_n|\, m(x) \\
&\leq \sup_{n\in\mathcal{N}} \Big(\sum_{x\in\mathcal{X}} (1+|x-n|)^{-d-\epsilon}\Big) \Big(\sup_{x\in\mathcal{X},n\in\mathcal{N}} (1+|x-n|)^{-s} m(n)^{-1} m(x)\Big) \qquad (8)\\
&\qquad \times \Big(\sum_{n\in\mathcal{N}} |c_n| m(n)\Big) \\
&= C\, \|c\|_{\ell^1_m(\mathcal{N})} .
\end{aligned}$$

The first supremum occurring in (8) is finite by Lemma 2.1, the second supremum is finite because of $m(x-n+n) \leq C'(1+|x-n|)^s m(n)$ for all $x, n \in \mathbb{R}^d$.

Boundedness on ℓ^∞.

$$\begin{aligned}
\|Ac\|_{\ell^\infty_m(\mathcal{X})} &= \sup_{x\in\mathcal{X}} \Big|\sum_{n\in\mathcal{N}} A_{xn}\, c_n\Big|\, m(x) \\
&\leq \sup_{x\in\mathcal{X}} \sum_{n\in\mathcal{N}} (1+|x-n|)^{-s-d-\epsilon}|c_n| m(x) \\
&\leq \Big(\sup_{x\in\mathcal{X}} \sum_{n\in\mathcal{N}} (1+|x-n|)^{-d-\epsilon}\Big) \Big(\sup_{x\in\mathcal{X},n\in\mathcal{N}} (1+|x-n|)^{-s} m(x) m(n)^{-1}\Big) \\
&\qquad \times \Big(\sup_{n\in\mathcal{N}} |c_n|\, m(n)\Big) \\
&\leq C\, \|c\|_{\ell^\infty_m(\mathcal{N})} .
\end{aligned}$$

(b) For sub-exponential weights we need a slight modification.

$$\begin{aligned}
\|Ac\|_{\ell^1_m(\mathcal{X})} &\leq \sum_{x\in\mathcal{X}}\sum_{n\in\mathcal{N}} e^{-\alpha|x-n|}|c_n|\, m(x) \\
&\leq \sup_{n\in\mathcal{N}}\Big(\sum_{x\in\mathcal{X}} e^{-\alpha|x-n|/2}\Big)\ \Big(\sup_{x\in\mathcal{X},n\in\mathcal{N}} e^{-\alpha|x-n|/2}m(n)^{-1}m(x)\Big) \\
&\qquad \times \Big(\sum_{n\in\mathcal{N}} |c_n| m(n)\Big) \\
&= C\,\|c\|_{\ell^1_m(\mathcal{N})}\,,
\end{aligned}$$

since $m(x-n+n) \leq C\, e^{\gamma|x-n|^\beta}\, m(n) \leq C' e^{\alpha|x-n|/2} m(n)$. The case $p=\infty$ is done similarly. ■

3. A Concept for the Localization of Frames

In this section we introduce an axiomatic theory of localized frames. After some set-up we define the localization of a frame with respect to a given Riesz basis and then investigate the properties of the dual frame.

3.1. **Associated Banach Spaces.** We first define an abstract class of Banach spaces associated to every orthonormal basis or Riesz basis of a Hilbert space. In concrete situations, these spaces turn out to be well-known function spaces from analysis, such as shift-invariant spaces (Section 5), Besov spaces, or modulation spaces (Section 4).

Let $\{g_n : n\in\mathcal{N}\}$ be a Riesz basis of $\mathcal{H}$ with dual basis $\{\widetilde{g_n} : n\in\mathcal{N}\}$. Let m be a weight function on $\mathbb{R}^d$ of polynomial or sub-exponential type.

Definition 1. Assume that $\ell^p_m(\mathcal{N}) \subseteq \ell^2(\mathcal{N})$. Then the Banach space $\mathcal{H}^p_m$ is defined to be

$$\mathcal{H}^p_m = \{f\in\mathcal{H} : f = \sum_{n\in\mathcal{N}} c_n g_n \text{ for } c\in\ell^p_m(\mathcal{N})\} \tag{9}$$

with norm $\|f\|_{\mathcal{H}^p_m} = \|c\|_{\ell^p_m}$.

Note that c_n is uniquely determined, in fact, $c_n = \langle f, \widetilde{g_n}\rangle$.

Since $\ell^p_m(\mathcal{N}) \subseteq \ell^2(\mathcal{N})$, then $\mathcal{H}^p_m$ is a (dense) subspace of $\mathcal{H}$. If $\ell^p_m \not\subseteq \ell^2$ and $p<\infty$, we define $\mathcal{H}^p_m$ to be the completion of the subspace $\mathcal{H}_0$ of finite linear combinations, i.e., $\mathcal{H}_0 = \{f = \sum_{n\in\mathcal{N}} c_n g_n : \operatorname{supp} c \text{ finite}\}$, with respect to the norm $\|f\|_{\mathcal{H}^p_m} = \|c\|_{\ell^p_m}$. If $p=\infty$ and $\ell^\infty_m \not\subseteq \ell^2$, we take the weak-* completion of $\mathcal{H}_0$ to define $\mathcal{H}^\infty_m$. Of course, in these cases $\mathcal{H}^p_m$ is no longer contained in $\mathcal{H}$.

In concrete situations with more structure, the series $f = \sum_{n\in\mathcal{N}} c_n g_n$ always converges in some super-space of distributions $\mathcal{D}' \supseteq \mathcal{H}$, even when $(c_n)_{n\in\mathcal{N}}$ is taken from a "large" sequence space. Then (9) make sense as a subspace of $\mathcal{D}'$ instead of $\mathcal{H}$. See Sections 4 and 5 for examples of associated Banach spaces $\mathcal{H}^p_m$ that are not contained in $\mathcal{H}$.

3.2. **Frames and Frame Operators.** Let $\mathcal{E} = \{e_x : x \in \mathcal{X}\}$ be a frame for $\mathcal{H}$ and $Sf := \sum_{x\in\mathcal{X}} \langle f, e_x\rangle e_x$ be the corresponding frame operator.

Each frame element has a natural expansion with respect to the given Riesz basis as follows:

$$e_x = \sum_{n\in\mathcal{N}} \langle e_x, \widetilde{g_n}\rangle g_n = \sum_{n\in\mathcal{N}} \langle e_x, g_n\rangle \widetilde{g_n}\,. \tag{10}$$

Since $\{e_x : x \in \mathcal{X}\}$ is a frame, the frame operator S is invertible on $\mathcal{H}$ by definition. Our main objective is to understand the mapping properties of S on the associated Banach spaces $\mathcal{H}^p_m$. For this purpose we study the frame operator with respect to the basis $\{g_n : n \in \mathcal{N}\}$.

We expand $f = \sum_n f_n g_n$ and e_x with respect to the Riesz basis and obtain that

$$\begin{aligned} Sf &= \sum_{x\in\mathcal{X}} \langle f, e_x\rangle e_x \\ &= \sum_{x\in\mathcal{X}} \sum_{n\in N} f_n \langle g_n, e_x\rangle e_x \\ &= \sum_{x\in\mathcal{X}} \sum_{m\in N} \sum_{n\in N} f_n \langle g_n, e_x\rangle \langle e_x, \widetilde{g_m}\rangle g_m \\ &= \sum_m \Big(\sum_n \Big(\sum_x \langle g_n, e_x\rangle \langle e_x, \widetilde{g_m}\rangle \Big) f_n \Big) g_m\,. \end{aligned}$$

Now let T be the infinite matrix with entries

$$T_{mn} = \sum_{x\in\mathcal{X}} \langle g_n, e_x\rangle \langle e_x, \widetilde{g_m}\rangle = \langle S g_n, \widetilde{g_m}\rangle\,, \tag{11}$$

and let Γ be the mapping

$$\Gamma : \mathcal{H} \mapsto \ell^2(\mathcal{N}), \qquad (\Gamma f)_n = \langle f, \widetilde{g_n}\rangle\,. \tag{12}$$

Since $\{g_n\}$ is a Riesz basis, Γ is invertible, and so the calculation above can be recast concisely as $S = \Gamma^{-1}\, T\, \Gamma$, or by means of a commutative diagram as

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{S} & \mathcal{H} \\ \Gamma\downarrow & & \downarrow\Gamma \\ \ell^2(\mathcal{N}) & \xrightarrow{T} & \ell^2(\mathcal{N})\,. \end{array} \tag{13}$$

This observation carries over to the Banach spaces $\mathcal{H}^p_m$, because Γ is an isometric isomorphism between $\mathcal{H}^p_m$ and $\ell^p_m(\mathcal{N})$. To understand the behavior of S on $\mathcal{H}^p_m$, it therefore suffices to study the matrix T on the sequence space $\ell^p_m(\mathcal{N})$. This task is conceptually much simpler.

For the investigation of T on $\ell^p_m(\mathcal{N})$ we will use a fundamental theorem of Jaffard [38]. We cite a version where the index set is a separated subset $\mathcal{N}$ of $\mathbb{R}^d$.

Theorem 3.1 (Jaffard's Theorem). *Assume that the matrix $G = (G_{kl})_{k,l\in\mathcal{N}}$ satisfies the following properties:*

(a) G is invertible as an operator on $\ell^2(\mathcal{N})$, and

(b) $|G_{kl}| \le C\,(1+|k-l|)^{-r}$, $k,l \in \mathcal{N}$ for some constant $C>0$ and some $r>d$.

Then the inverse matrix $H = G^{-1}$ satisfies the same off-diagonal decay, that is,

$$|H_{kl}| \le C'\,(1+|k-l|)^{-r}, \quad k,l \in \mathcal{N}\,. \tag{14}$$

The proof of Theorem 3.1 is rather delicate (see [38] for the detailed argument). Similar statements for different conditions on the off-diagonal decay can be found in [9].

For exponential decay off the diagonal, we use the following theorem, see [38,42].

Theorem 3.2. *Assume that the matrix $G = (G_{kl})_{k,l\in\mathcal{N}}$ satisfies the following properties:*

(a) G is invertible as an operator on $\ell^2(\mathcal{N})$, and

(b) $|G_{kl}| \le C\,e^{-\alpha|k-l|}$, $k,l \in \mathcal{N}$ for some constants $\alpha, C > 0$.

Then there exists $\alpha' \in (0,\alpha)$ such that the inverse matrix $H = G^{-1}$ satisfies the off-diagonal decay of the form

$$|H_{kl}| \le C'\,e^{-\alpha'|k-l|}, \quad k,l \in \mathcal{N}\,. \tag{15}$$

3.3. **Localization of Frames.** After these preparations we now can introduce the main concept of this paper, a new notion of localization for frames.

Definition 2. We say that the frame $\mathcal{E} = \{e_x : x \in \mathcal{X}\}$ is polynomially localized with respect to the Riesz basis $\{g_n\}$ with decay $s>0$ (or simply *s-localized*), if

$$|\langle e_x, g_n\rangle| \le C(1+|x-n|)^{-s} \tag{16}$$

and

$$|\langle e_x, \widetilde{g_n}\rangle| \le C(1+|x-n|)^{-s} \tag{17}$$

for all $n \in \mathcal{N}$ and $x \in \mathcal{X}$.

Likewise $\mathcal{E}$ is called *exponentially localized*, if for some $\alpha > 0$

$$|\langle e_x, g_n\rangle| \le Ce^{-\alpha|x-n|} \qquad \text{and} \qquad |\langle e_x, \widetilde{g_n}\rangle| \le Ce^{-\alpha|x-n|}\,.$$

REMARKS: 1. Clearly, the localization of a frame depends on the given Riesz basis. One may define an equivalence relation for Riesz bases and then show that the localization properties depend only on the equivalence class of the Riesz basis. For simplicity, we indicate this for orthonormal bases only. We call two orthonormal bases $\{g_n : n \in \mathcal{N}\}$ and $\{h_m : m \in \mathcal{N}'\}$ s-equivalent, if

$$|\langle g_n, h_m\rangle| \le C(1+|n-m|)^{-s} \qquad \forall n \in \mathcal{N}, m \in \mathcal{N}'\,.$$

Using Lemma 2.2, it is easy to see that a frame $\mathcal{E}$ is s-localized with respect to the orthonormal basis $\{g_n\}$ if and only if it is s-localized with respect to $\{h_m\}$.

2. Conditions (16) and (17) coincide for orthonormal bases, but they are independent for general Riesz bases. Only for certain Riesz bases these conditions are equivalent. It can be shown that (17) is implied by (16) and viceversa whenever the Gram matrix of the Riesz basis satisfies the conditions $|\langle g_n, g_m\rangle| \le C(1+|m-n|)^{-s}$, $m,n \in \mathcal{N}$. As the explicit computation and estimation of Riesz bases can be very cumbersome, one will mostly use the localization of frames with respect to an orthonormal

basis. In Section 4, however, we will work with the full definition and use Riesz bases.

We first show that the frame operator of localized frames is well behaved on the Banach spaces $\mathcal{H}^p_m$.

Proposition 3.3. *Given $1 \le p \le \infty$, $s \ge 0$, and an s-moderate weight m.*
Assume that $\mathcal{E}$ is an $(s+d+\epsilon)$-localized frame for some $\epsilon > 0$.
(a) Then the coefficient operator defined by $C_{\mathcal{E}} f = \big(\langle f, e_x\rangle\big)_{x\in\mathcal{X}}$ is bounded from $\mathcal{H}^p_m$ to $\ell^p_m(\mathcal{X})$.
(b) The synthesis operator defined on finite sequences by $D_{\mathcal{E}} c = \sum_{x\in\mathcal{X}} c_x e_x$ extends to a bounded mapping from $\ell^p_m(\mathcal{X})$ to $\mathcal{H}^p_m$.
(c) The frame operator $S = S_{\mathcal{E}} = D_{\mathcal{E}} C_{\mathcal{E}} = \sum_{x\in\mathcal{X}} \langle f, e_x\rangle e_x$ maps $\mathcal{H}^p_m$ into $\mathcal{H}^p_m$, and the series converges unconditionally for $1 \le p < \infty$.
If $\mathcal{E}$ is an exponentially localized frame , then these statements hold for all subexponential weights.

Proof. We prove these statements for $(s+d+\epsilon)$-localized frames. For exponentially localized frames we use Lemma 2.3(b) instead of (a).

(a) Let $f = \sum_{n\in\mathcal{N}} f_n g_n$, then the localization estimates (16) imply that

$$\begin{aligned} |\langle f, e_x\rangle| &= \Big|\sum_{n\in\mathcal{N}} f_n \langle g_n, e_x\rangle\Big| \\ &\le C \sum_{n\in\mathcal{N}} |f_n| \big(1 + |x-n|\big)^{-s-d-\epsilon} \end{aligned}$$

If $f \in \mathcal{H}^p_m$, then $\|f\|_{\mathcal{H}^p_m} = \|(f_n)_{n\in\mathcal{N}}\|_{\ell^p_m(\mathcal{N})}$, and Lemma 2.3(a) implies that

$$\|C_{\mathcal{E}} f\|_{\ell^p_m(\mathcal{X})} \le C \|(f_n)_{n\in\mathcal{N}}\|_{\ell^p_m(\mathcal{N})} = C\|f\|_{\mathcal{H}^p_m}\,.$$

(b) We need to show that the sequence with entries $(D_{\mathcal{E}} c)_n = \langle \sum_{x\in\mathcal{X}} c_x e_x, \tilde{g}_n\rangle$ is in $\ell^p_m(\mathcal{N})$. Using (17) and the notation of Lemma 2.3(a), we obtain that

$$\begin{aligned} |(D_{\mathcal{E}} c)_n| &\le \sum_{x\in\mathcal{X}} |c_x|\, |\langle e_x, \tilde{g}_n\rangle| \\ &\le C \sum_{x\in\mathcal{X}} |c_x|\, (1+|x-n|)^{-s-d-\epsilon} = C\,(A^*|c|)_n\,. \end{aligned}$$

Now Lemma 2.3(a) (with $\mathcal{N}$ and $\mathcal{X}$ interchanged) imply the boundedness of $D_{\mathcal{E}}$:

$$\|D_{\mathcal{E}} c\|_{\mathcal{H}^p_m} = \|A^*|c|\|_{\ell^p_m(\mathcal{N})} \le \|A^*\|_{op}\, \|c\|_{\ell^p_m(\mathcal{X})}\,.$$

(c) The boundedness of $S = D_{\mathcal{E}} C_{\mathcal{E}}$ follows by combining (a) and (b). As for the unconditional convergence of the series defining S, let $\epsilon > 0$ and choose $\mathcal{N}_0 = \mathcal{N}_0(\epsilon)$, such that $\|\langle f, e_x\rangle_{x\notin\mathcal{N}_0}\|_{\ell^p_m} \le \epsilon$. Then for any finite set $\mathcal{N}_1 \supseteq \mathcal{N}_0$, assertions (a) and (b) imply that

$$\|Sf - \sum_{x\in\mathcal{N}_1} \langle f, e_x\rangle e_x\|_{\mathcal{H}^p_m} \;\le\; \|C_{\mathcal{E}}\|_{op}\, \|\langle f, e_x\rangle_{x\notin\mathcal{N}_1}\|_{\ell^p_m} < \|C_{\mathcal{E}}\|_{op}\,\epsilon\,.$$

This means that $\sum_{x\in\mathcal{X}} \langle f, e_x\rangle e_x$ converges unconditionally in $\mathcal{H}^p_m$. ■

REMARK: Note that the proof shows that the operator norms of $C_{\mathcal{E}}, D_{\mathcal{E}}$ and $S_{\mathcal{E}}$ can bounded uniformly by a constant that depends only on s, but not on m or p.

Next we investigate how the localization of a frame affects the system matrix defined in (11).

Proposition 3.4. *(a) Assume that $\mathcal{E} = \{e_x : x \in \mathcal{X}\}$ is polynomially localized with respect to the Riesz basis $\{g_n\}$ with decay $s > d$. Then*

$$|T_{mn}| \le C(1 + |m-n|)^{-s} \qquad \text{for } m, n \in \mathcal{N}\,. \tag{18}$$

(b) If $\mathcal{E}$ is exponentially localized, then for some $\alpha' > 0$

$$|T_{mn}| \le C e^{-\alpha'|m-n|}, \qquad \text{for } m, n \in \mathcal{N}\,. \tag{19}$$

Proof. (a) We insert the inequalities (16) and (17) into (11) and obtain

$$|T_{mn}| \le C \sum_{x \in \mathcal{X}} \Big((1 + |x-n|)^{-s} (1 + |x-m|)^{-s} \Big)\,. \tag{20}$$

Now apply Lemma 2.2(a).

(b) is proved similarly by using Lemma 2.2(b) ■

3.4. **Localization of the Dual Frame.** With these concepts we are now ready to prove the main theorem about the localization of frames and their dual frames. Recall that d is the dimension of the "carrier" space $\mathbb{R}^d$ and that all index sets $\mathcal{N}$ and $\mathcal{X}$ are subsets of $\mathbb{R}^d$. For polynomial decay the conditions depend on d.

Theorem 3.5. *Assume that $\{e_x : x \in \mathcal{X}\}$ is a frame with polynomial decay $s+d+\epsilon$ with respect to the Riesz basis $\{g_n\}$ for some $\epsilon > 0$.*

(a) Then the frame operator S is invertible simultaneously on all Banach spaces $\mathcal{H}^p_m$, where $1 \le p \le \infty$ and m is an s-moderate weight.

(b) The dual frame $\{\widetilde{e_x} = S^{-1} e_x : x \in \mathcal{X}\}$ is polynomially localized with the same decay $s + d + \epsilon$.

(c) The frame expansion

$$f = \sum_{x \in \mathcal{X}} \langle f, e_x \rangle \widetilde{e_x} = \sum_{x \in \mathcal{X}} \langle f, \widetilde{e_x} \rangle e_x \tag{21}$$

converges unconditionally in $\mathcal{H}^p_m$ for $1 \le p < \infty$ (and weak unconditionally in $\mathcal{H}^\infty_m$).*

(d) We have the norm equivalence

$$\|f\|_{\mathcal{H}^p_m} \asymp \Big(\sum_{x \in \mathcal{X}} |\langle f, e_x \rangle|^p m(x)^p \Big)^{1/p} \asymp \Big(\sum_{x \in \mathcal{X}} |\langle f, \widetilde{e_x} \rangle|^p m(x)^p \Big)^{1/p} \tag{22}$$

We can rephrase part (b) of the theorem by saying that *the dual frame possesses the same localization properties as the original frame.*

Proof. (a) Consider the matrix T defined in (11). By Proposition 3.4 the entries of T have the following off-diagonal decay:

$$T_{mn} = \mathcal{O}((1 + |m-n|)^{-s-d-\epsilon})\,. \tag{23}$$

Since S acting on $\mathcal{H}$ and T acting on $\ell^2(\mathcal{N})$ are conjugate by (13), they have the same spectrum. By assumption S is invertible, therefore T is also invertible. These are the hypotheses of Theorem 3.1 (with $r = s + d + \epsilon > d$), and so we conclude that the entries of the inverse matrix also satisfy

$$(T^{-1})_{mn} \leq C(1 + |m - n|)^{-s-d-\epsilon} \tag{24}$$

By Lemma 2.3(a) (with $\mathcal{N} = \mathcal{X}$), T^{-1} extends to a bounded operator on all sequence spaces $\ell^p_m(\mathcal{N})$ simultaneously for all s-moderate weight functions m and all $p \in [1, \infty]$. By the diagram (13) S is then invertible on $\mathcal{H}^p_m$.

(b) We have to check conditions (16) and (17) for the dual frame $\{\widetilde{e_x}\}$. By (13) and (24) we find that

$$|(T^{-1})_{mn}| = |\langle S^{-1} g_n, \widetilde{g_m} \rangle| \leq C\big(1 + |m - n|\big)^{-s-d-\epsilon}.$$

Using the biorthogonal expansion (10) of e_x, we obtain

$$\begin{aligned} \langle \widetilde{e_x}, g_n \rangle &= \langle S^{-1} e_x, g_n \rangle = \langle e_x, S^{-1} g_n \rangle \\ &= \sum_{m \in \mathcal{N}} \langle e_x, g_m \rangle \, \langle \widetilde{g_m}, S^{-1} g_n \rangle \\ &= \sum_{m \in \mathcal{N}} \langle e_x, g_m \rangle \, \overline{(T^{-1})_{mn}}\,. \end{aligned} \tag{25}$$

Estimate (16) for $\widetilde{e_x}$ follows by means of Lemma 2.2(a)

$$\begin{aligned} |\langle \widetilde{e_x}, g_n \rangle| &\leq C_1 \sum_{m \in \mathcal{N}} \big(1 + |x - m|\big)^{-s-d-\epsilon} \big(1 + |m - n|\big)^{-s-d-\epsilon} \\ &\leq C_2 \big(1 + |x - n|\big)^{-s-d-\epsilon}. \end{aligned}$$

The second estimate (17) follows similarly from

$$\begin{aligned} \langle \widetilde{e_x}, \widetilde{g_n} \rangle &= \langle e_x, S^{-1} \widetilde{g_n} \rangle \\ &= \sum_{m \in \mathcal{N}} \langle e_x, \widetilde{g_m} \rangle \, \langle g_m, S^{-1} \widetilde{g_n} \rangle = \sum_{m \in \mathcal{N}} \langle e_x, \widetilde{g_m} \rangle \, (T^{-1})_{nm}\,. \end{aligned}$$

(c) Since the series $\sum_{x \in \mathcal{X}} \langle f, e_x \rangle e_x$ converges unconditionally by Prop. 3.3(c), the series $S^{-1}(\sum_{x \in \mathcal{X}} \langle f, e_x \rangle e_x)$ also converges unconditionally, see also [33, Lemma 5.3.2].

(d) Since $f = S^{-1} S f = S^{-1}(D_{\mathcal{E}} C_{\mathcal{E}} f)$, we have

$$\|f\|_{\mathcal{H}^p_m} \leq \|S^{-1}\|_{op} \|D_{\mathcal{E}}\|_{op} \|C_{\mathcal{E}} f\|_{\ell^p_m(\mathcal{X})} \leq \|S^{-1}\|_{op} \|D_{\mathcal{E}}\|_{op} \|C_{\mathcal{E}}\|_{op} \|f\|_{\mathcal{H}^p_m}\,.$$

The second norm equivalence is shown in the same way by using the fact that $S^{-1} f = \sum_{x \in \mathcal{X}} \langle f, \widetilde{e_x} \rangle \widetilde{e_x}$ (e.g., [33, Lemma 5.1.6]). ■

We now formulate a version of the main theorem for exponentially localized frames. In this case the class of admissible weights can be extended to arbitary sub-exponential weights.

Theorem 3.6. *Assume that $\mathcal{E} = \{e_x : x \in \mathcal{X}\}$ is an exponentially localized frame with respect to the Riesz basis $\{g_n\}$.*

(a) Then the frame operator S is invertible simultaneously on all Banach spaces $\mathcal{H}^p_m$ for all sub-exponential weight functions m.

(b) The dual frame $\{\widetilde{e_x} : x \in \mathcal{X}\}$ is also exponentially localized.

Furthermore, the frame expansion (21) converges unconditionally in $\mathcal{H}^p_m$ for $1 \leq p < \infty$ and all sub-exponential weights m, and the norm equivalence (22) holds.

Proof. The proof is similar to the proof of Theorem 3.5. We apply Proposition 3.4(b) and see that Theorem 3.2 implies the existence of $\alpha' < \alpha$ such that $|(T^{-1})_{mn}| \leq Ce^{-\alpha'|m-n|}$ for $m, n \in \mathcal{N}$. Consequently, T^{-1} is bounded on all $\ell^p_m(\mathcal{N})$ with a sub-exponential weight m by Lemma 2.3(b). The rest follows as above. ∎

3.5. **Banach Frames.** Banach frames generalize the concept of frames for Hilbert spaces to Banach spaces [30]. We investigate the statements of Theorems 3.5 and 3.6 in the light of Banach frames.

Definition 3. A *Banach frame* of a separable Banach space B is a countable set $\mathcal{E} = \{e_x : x \in \mathcal{X}\} \subseteq B'$ with an associated sequence space B_d such that the following properties hold.

(a) The coefficient operator $C_\mathcal{E}$ defined by $C_\mathcal{E} f = \Big(\langle f, e_x\rangle_{x\in\mathcal{X}}\Big)$ is bounded from B into B_d.

(b) Norm equivalence:

$$\|f\|_B \asymp \|\langle f, e_x\rangle_{x\in\mathcal{X}}\|_{B_d} \, . \tag{26}$$

(c) There exists a bounded operator R from B_d onto B, a so-called reconstruction operator, such that

$$R\Big(\langle f, e_x\rangle_{x\in\mathcal{X}}\Big) = f \, . \tag{27}$$

In other words, $RC_\mathcal{E} = \mathrm{I}_B$, and the following diagram commutes:

$$\begin{array}{ccc} & & B_d \\ & C \nearrow & \downarrow R \\ B & \xrightarrow{\mathrm{I}} & B \end{array}$$

We can now reformulate Theorems 3.5 and 3.6 as follows.

Theorem 3.7. *Assume that $\mathcal{E} = \{e_x : x \in \mathcal{X}\}$ is a localized frame for the Hilbert space $\mathcal{H}$. Then $\mathcal{E}$ is automatically a Banach frame for $\mathcal{H}^p_m, 1 \leq p \leq \infty$.*

Specifically, if $\mathcal{E}$ is polynomially localized with decay $s + d + \epsilon, \epsilon > 0$, then it is a Banach frame for all $\mathcal{H}^p_m$ for all s-moderate weight functions m. If $\mathcal{E}$ is exponentially localized, then it is a Banach frame for all $\mathcal{H}^p_m$ for all sub-exponential weight functions m.

Moreover, the reconstruction operator R coincides with the inverse frame operator, that is, we can take $R = S^{-1}$.

Proof. This is just a reformulation of Theorems 3.5 and 3.6 in a different terminology. ■

To summarize, localized frames are automatically Banach frames. They are *universal* in the sense that they form a Banach frame simultaneously for the entire family of associated Banach spaces $\mathcal{H}^p_m$.

REMARKS: 1. See [30] for the original definition of Banach frames and [7,13,15,27] for subsequent work. The atomic decompositions mentioned in the introduction can be interpreted as constructions of Banach frames [17, 29].

2. Some authors use a weaker concept instead of Definition 3. A set $\mathcal{E}$ is called a p-frame, if only conditions (a) and (b) hold with $B_d = \ell^p$ [7].

For localized frames, the two concepts of p-frames and Banach frames coincide by Theorem 3.7. In addition, Theorem 3.7 provides an explicit reconstruction operator R, namely the inverse frame operator.

4. Non-Uniform Sampling in Shift-Invariant Spaces

As our first application we discuss the (non-uniform) sampling problem in shift-invariant spaces of $\mathbb{R}^d$. For a detailed exposition and a comprehensive list of references we refer to [6].

4.1. **Shift-Invariant Spaces.** Let $T_x f(t) = f(t-x), t, x \in \mathbb{R}^d$, be the translation operator. Choose a so-called generator $\varphi \in L^2(\mathbb{R}^d)$ and define the shift-invariant space space $V^p_m(\varphi)$ as a subspace of L^p_m by

$$V^p_m(\varphi) = \{f \in L^p_m(\mathbb{R}^d) : f = \sum_{k\in\mathbb{Z}^d} c_k T_k \varphi \text{ with } c \in \ell^p_m(\mathbb{Z}^d)\} \tag{28}$$

For this definition to make sense, we impose the following standard assumptions on the generator φ:

(i) The translates $\{T_k\varphi : k \in \mathbb{Z}^d\}$ form a Riesz basis for the Hilbert space $V^2(\varphi)$. We also say that "φ is a stable generator".

(ii) φ is continuous.

(iii) To deal with s-moderate weights, we assume that φ satisfies the decay condition

$$|\varphi(x)| \le C(1+|x|)^{-d-s-\epsilon} \quad \text{for some } \epsilon > 0\,. \tag{29}$$

(iii') To deal with sub-exponential weight functions, we assume that φ decays exponentially as

$$|\varphi(x)| \le Ce^{-\alpha|x|}\,. \tag{30}$$

Under the assumptions (i) — (iii) or (iii') the sum in (28) converges unconditionally in $V^p_m(\varphi)$ and uniformly [6], and $V^p_m(\varphi)$ is a closed subspace of $L^p_m(\mathbb{R}^d)$ endowed with the equivalent norms

$$\|f\|_{L^p_m} \asymp \|c\|_{\ell^p_m}\,. \tag{31}$$

Comparing (28) with (9), we see that the shift-invariant space $V^p_m(\varphi)$ coincides with the Banach space $\mathcal{H}^p_m$ associated to the Riesz basis $\{T_k\varphi\}$.

4.2. **The Dual Generator.** Since the Riesz basis $\{T_k\varphi : k \in \mathbb{Z}^d\}$ is invariant under translations, the dual basis must again be of the form $\{T_k\tilde{\varphi} : k \in \mathbb{Z}^d\}$. The dual generator $\tilde{\varphi}$ satisfies the relations

$$\langle T_k\tilde{\varphi}, T_l\varphi\rangle = \delta_{kl} \tag{32}$$

and every $f \in V^2(\varphi)$ possesses the biorthogonal expansion

$$f = \sum_{k\in\mathbb{Z}^d} \langle f, T_k\tilde{\varphi}\rangle T_k\varphi\,. \tag{33}$$

To apply Theorems 3.5 and 3.6, we need an estimate for the decay of $\tilde{\varphi}$.

Lemma 4.1. *(a) If φ is stable and satisfies $|\varphi(x)| \le C(1+|x|)^{-r}$ for some $r > d$, then the dual generator satisfies*

$$|\tilde{\varphi}(x)| \le C'(1+|x|)^{-r}\,.$$

(b) If φ is stable and decays exponentially, i.e., $|\varphi(x)| \le Ce^{-\alpha|x|}$, then $|\tilde{\varphi}(x)| \le C'e^{-\alpha'|x|}$ for some $\alpha', 0 < \alpha' < \alpha$.

Proof. (a) Since $\tilde{\varphi} \in V^2(\varphi)$, it possesses the series expansion

$$\tilde{\varphi} = \sum_{k\in\mathbb{Z}^d} b_k T_k\varphi\,. \tag{34}$$

The coefficients b_n are determined by the biorthogonality condition (32):

$$\begin{aligned}
\delta_l &= \langle\tilde{\varphi}, T_l\varphi\rangle \\
&= \sum_{m\in\mathbb{Z}^d} b_m\langle T_m\varphi, T_l\varphi\rangle \\
&= \sum_{m\in\mathbb{Z}^d} b_m\langle\varphi, T_{l-m}\varphi\rangle
\end{aligned}$$

This convolution can be written with the (infinite) matrix Φ with entries $\Phi_{lm} = \langle\varphi, T_{l-m}\varphi\rangle = \gamma_{l-m}$. The assumption on the decay of φ and Lemma 2.2(a) imply that

$$|\Phi_{lm}| \le C(1+|l-m|)^{-r} \quad l,m \in \mathbb{Z}^d\,.$$

Since $\{T_k\varphi : k \in \mathbb{Z}^d\}$ is a Riesz basis for $V^2(\varphi)$, its Gram matrix Φ is invertible on $\ell^2(\mathbb{Z}^d)$, and Φ^{-1} is again a convolution with a sequence, say β. Since $r > d$, Theorem 3.1 yields the decay estimate

$$|(\Phi^{-1})_{lm}| = |\beta_{l-m}| \le C'(1+|l-m|)^{-r}$$

Consequently

$$b = \Phi^{-1}\delta = \beta * \delta = \beta\,.$$

Thus

$$|b_l| \le C(1+|l|)^{-r} \quad l \in \mathbb{Z}^d\,,$$

and invoking Lemma 2.2(a) once again we obtain that $|\tilde{\varphi}(x)| \le C(1+|x|)^{-r}$.

(b) The proof is similar. Just use Theorem 3.2 and Lemma 2.2(b). ■

REMARK: The above proof emphasizes the role of Theorems 3.1 and 3.2. Alternative proofs can be based on the group structure of the index set $\mathbb{Z}^d$ and use Fourier series, Wiener's Lemma, and analyticity. See [6,18] for arguments in this direction.

4.3. **Sampling Theorems.** As a consequence of assumptions (i) — (iii) and (iii'), every pointwise evaluation $f \to f(x)$ is a continuous linear functional on $\mathcal{H}^p_m$. For $p = 2$ and $m \equiv 1$ this implies that there exist kernel functions $K_x \in V^2(\varphi)$ such that

$$f(x) = \langle f, K_x \rangle \,. \tag{35}$$

In the usual Hilbert space setting, a sampling theorem of the form

$$\sum_{x \in \mathcal{X}} |f(x)|^2 \asymp \|f\|_2 \tag{36}$$

is therefore equivalent to saying that the set $\{K_x : x \in \mathcal{X}\}$ is a frame for $V^2(\varphi)$ [1,10]. If (36) is satisfied for $\mathcal{X} \subseteq \mathbb{R}^d$, $\mathcal{X}$ is called a *set of sampling* for $V^2(\varphi)$.

Our aim is to extend (36) to a sampling theorem in $V^p_m(\varphi)$ and to derive sampling inequalities of the form

$$A\|f\|_{L^p_m} \le \Big(\sum_{x \in \mathcal{X}} |f(x)|^p m(x)^p \Big)^{1/p} \le B\|f\|_{L^p_m} \qquad \text{for } f \in V^p_m(\varphi) \,. \tag{37}$$

In other words, we want to construct a Banach frame of the form $\{K_x : x \in \mathcal{X}\}$ for $V^p_m(\varphi)$. To apply the abstract theory of Section 3, we need to check conditions (16) and (17) for a frame of the form $\{K_x : x \in \mathcal{X}\}$.

Using the decay properties of the generators φ and $\tilde{\varphi}$ (Lemma 4.1 with $r = s + d + \epsilon$), and (35), we estimate

$$|\langle K_x, T_k \varphi \rangle| = |\varphi(x-k)| \le C(1 + |x-k|)^{-s-d-\epsilon}$$

and

$$|\langle K_x, T_k \tilde{\varphi} \rangle| = |\tilde{\varphi}(x-k)| \le C(1 + |x-k|)^{-s-d-\epsilon} \,.$$

Consequently $\{K_x : x \in \mathcal{X}\}$ is $(s+d+\epsilon)$-localized. We have now verified all hypotheses of Theorem 3.5, and thus we can deduce the following sampling theorem in shift-invariant spaces.

Theorem 4.2. *Assume that the generator φ satisfies the assumptions (i) — (iii), that m is s-moderate, and that $\mathcal{X}$ is a set of sampling for $V^2(\varphi)$ with dual frame $\widetilde{K}_x$.*

(a) Then we have for every $f \in V^p_m(\varphi), 1 \le p \le \infty$, that

$$A\|f\|_{L^p_m} \le \Big(\sum_{x \in \mathcal{X}} |f(x)|^p m(x)^p \Big)^{1/p} \le B\|f\|_{L^p_m} \,. \tag{38}$$

(b) Each $\widetilde{K}_x$ satisfies the localization estimate

$$|\widetilde{K}_x(t)| \le C(1 + |t-x|)^{-s-d-\epsilon} \qquad \textit{for all } x \in \mathcal{X}, t \in \mathbb{R}^d \,, \tag{39}$$

with a constant C independent of x.

(c) The reconstruction series

$$f = \sum_{x \in \mathcal{X}} f(x) \widetilde{K}_x \tag{40}$$

converges unconditionally in $V^p_m(\varphi)$ for $1 \le p < \infty$.

Proof. Assertions (a) and (c) follow from Theorem 3.5. To prove (b), we write $\widetilde{K}_x$ as follows:

$$\widetilde{K}_x(t) = \langle \widetilde{K}_x, K_t \rangle = \sum_{k \in \mathbb{Z}^d} \langle \widetilde{K}_x, T_k \varphi \rangle \, \langle T_k \tilde{\varphi}, K_t \rangle \, .$$

Now use the localization estimate $|\langle \widetilde{K}_x, T_k \varphi \rangle| \le C(1 + |x-k|)^{-s-d-\epsilon}$, which is guaranteed by Theorem 3.5(b), the localization of K_x, and Lemma 2.2(a). ■

The frame reconstruction (40) is not only valid in $V^2(\varphi)$, but also holds in all $V^p_m(\varphi)$. If $V^p_m(\varphi) \subseteq V^2(\varphi)$, then (40) converges in a finer norm. This property can be seen as a very strong and useful form of stability of the frame $\{K_x : x \in \mathcal{X}\}$.

Note that $\widetilde{K}_x$ has its essential support in a neighborhood of x, therefore the reconstruction (40) is local in the sense that for the approximation of $f(x_0)$ we need only samples $f(x)$ for $x \in \mathcal{X}$ near x_0.

A similar theorem can be proved for exponentially decaying generators.

Theorem 4.3. *Assume that φ satisfies conditions (i) — (iii') and that $\mathcal{X}$ is a set of sampling for $V^2(\varphi)$ with dual frame $\widetilde{K}_x$. Then* (38) *and* (40) *hold for the shift-invariant spaces $V^p_m(\varphi)$ with sub-exponential weight m. There exists $\alpha' \in (0, \alpha)$, such that*

$$|\widetilde{K}_x(t)| \le C' e^{-\alpha' |t-x|} \qquad \text{for all } x \in \mathcal{X} \, . \tag{41}$$

REMARKS: 1. It is worth emphasizing that an L^2-estimate of the form $\|f\|_2 \asymp \left(\sum_{x \in \mathcal{X}} |f(x)|^2\right)^{1/2}$ for $f \in V^2(\varphi)$ implies *automatically* a weighted L^p-estimate of the form $\|f\|_{V^p_m(\varphi)} \asymp \Big(\sum_{x \in \mathcal{X}} |f(x)|^p m(x)^p\Big)^{1/p}$ for $f \in V^p_m(\varphi)$.

2. In contrast to sampling theorems in [3,6], the sampling density in all $V^p_m(\varphi)$ is determined entirely by the required density in the Hilbert space $V^2(\varphi)$ (and suitable decay of φ). The Hilbert space theory is well-understood and for a large class of generators and sampling sets $\mathcal{X}$, sharp sampling theorems of the form (36) have already been derived and are being used [5,6]. The reconstruction in the associated Banach spaces and the stability of this reconstruction then follow automatically. Theorem 4.2 solves a conjecture posed in [6].

3. Theorem 4.2 fails for band-limited functions. In this case the generator is $\varphi(t) = \frac{\sin \pi t}{\pi t}$ and $V^2(\varphi) = \{f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subseteq [-1/2, 1/2]\}$. We note that φ fails to satisfy the decay condition (29). While the conclusion of Theorem 4.2(a) is still valid for $V^p_m(\varphi) = V^2(\varphi) \cap L^p_m$ for $1 \le p \le 2$ and suitable weights, statements (b) and (c) are false: the dual frame lacks localization and the reconstruction (40) fails to converge in V^1_m. This counter-example shows that Theorem 3.5 is almost sharp.

4.4. **Sampling from Averages.** A number of recent papers emphasize that a realistic sampling model should use local averages $\langle f, \psi_j \rangle = \int_{\mathbb{R}^d} f(t)\psi_j(t)\,dt$ in place of exact pointwise values $f(x_j)$ as the input for reconstruction [2, 4, 31]. To interpret $\langle f, \psi_j \rangle$ as a *local* average (where the averaging procedure ψ_j may vary from point to point), we assume that

$$\sup_{j\in J} \|\psi_j\|_1 < \infty, \tag{42}$$

$$\int_{\mathbb{R}^d} \psi_j(t)\,dt = 1\,, \tag{43}$$

$$\operatorname{supp}\psi_j \subseteq x_j + [-a, a]^d\,. \tag{44}$$

As an application of the abstract theory of localized frames we prove the following theorem for sampling from averages. Again, it suffices to understand the L^2-theory in order to do sampling in shift-invariant spaces.

Theorem 4.4. *Assume that the generator φ satisfies conditions (i) — (iii) or (iii') and that the averaging functions ψ_j satisfy* (42)—(44).

If

$$A\|f\|_2^2 \le \sum_{j\in J} |\langle f, \psi_j\rangle|^2 \le B\|f\|_2^2 \tag{45}$$

holds for all $f \in V^2(\varphi)$, then the norm equivalence

$$A\ \|f\|_m^p \le \Big(\sum_{j\in J} |\langle f, \psi_j\rangle|^p m(x_j)^p\Big)^{1/p} \le B\ \|f\|_m^p \tag{46}$$

holds for all $f \in V_m^p(\varphi)$, where m can be an arbitrary s-moderate weights m if (29) *holds, and a sub-exponential weight if* (30) *holds.*

Proof. Let $\Psi_j \in V^2(\varphi)$ be the orthogonal projection of ψ_j onto $V^2(\varphi)$. Then

$$\langle f, \psi_j\rangle = \langle f, \Psi_j\rangle$$

and by (45) the set $\{\Psi_j : j \in J\}$ is a frame for $V^2(\varphi)$. We need to verify that this frame is localized. Inequality (16) turns into the estimate

$$\begin{aligned}
|\langle \Psi_j, T_k\varphi\rangle| &= |\langle \psi_j, T_k\varphi\rangle| = \Big|\int_{\mathbb{R}^d} \psi_j(t)\varphi(t-k)\,dt\Big| \\
&\le \int_{x_j+[-a,a]^d} |\psi_j(t)|(1+|t-k|)^{-s-d-\epsilon}\,dt \\
&\le \int_{x_j+[-a,a]^d} |\psi_j(t)|\,dt \sup_{t\in x_j+[-a,a]^d} (1+|t-k|)^{-s-d-\epsilon} \\
&\le C(1+|x_j-k|)^{-s-d-\epsilon}\,.
\end{aligned}$$

The estimate for $\langle \Psi_j, T_k\tilde{\varphi}\rangle$ is similar using Lemma 4.1.

Thus we have shown that $\{\Psi_j\}$ is a localized frame for $V^2(\varphi)$ and an application of Theorem 3.5 yields the conclusion. ■

5. Uniform and Non-Uniform Gabor Frames

Next we apply the theory of localized frames to time-frequency analysis. Write

$$T_xf(t) = f(t-x) \quad \text{and} \quad M_\omega f(t) = e^{2\pi i\omega\cdot t}f(t) \tag{47}$$

for the translation and modulation operators. The combination

$$\pi(z) = M_\omega T_x \qquad \text{for } z=(x,\omega)\in\mathbb{R}^{2d} \tag{48}$$

is called a time-frequency shift.

5.1. **Wilson Bases and Modulation Spaces.** By the Balian-Low theorem, regular sets of time-frequency shifts of a well-localized smooth function can never form a Riesz basis (see [33, Thm. 8.4.1] and the given references), but a remarkable trick of Daubechies, Jaffard, and Journée [20] allows us to construct orthonormal bases for $L^2(\mathbb{R})$ with a transparent time-frequency structure. These so-called Wilson bases are the natural bases in order to test Gabor frames for localization.

Given $\psi\in L^2(\mathbb{R})$, we define a Wilson system to be the following collection of functions: If $l=0$, set $\psi_{k0}=T_{\frac{k}{2}}\psi$, and for $l>0, l,k\in\mathbb{Z}$ we set

$$\psi_{kl} = \frac{1}{\sqrt{2}}\big(M_l + (-1)^{k+l}M_{-l}\big)\,T_{\frac{k}{2}}\psi\,. \tag{49}$$

It can be shown that there exist functions ψ such that (1) both ψ and $\hat{\psi}$ possess exponential decay and (2) the set $\{\psi_{kl}:(k,l)\in\mathbb{Z}\times\mathbb{N}_0\}$ is an orthonormal basis for $L^2(\mathbb{R})$ [20]. To extend this construction to $\mathbb{R}^d$, we use tensor products. Let $k=(k_1,\dots,k_d)\in\mathbb{Z}^d$ and $=(l_1,\dots,l_d)\in\mathbb{N}_0^d$, and set

$$\psi_{kl}(t_1,\dots,t_d) = \prod_{j=1}^d \psi_{k_j,l_j}(t_j)\,.$$

Then the set $\{\psi_{kl}:k\in\mathbb{Z}^d, l\in\mathbb{N}_0^d\}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$.

The associated Banach spaces $\mathcal{H}^p_m$ are known as modulation spaces and we define them in more generality as follows.

Definition 4. Let m be an s-moderate weight function. A tempered distribution $f\in\mathcal{S}'(\mathbb{R}^d)$ belongs to the modulation space $M^{p,q}_m$, if f possesses a Wilson expansion of the form

$$f = \sum_{(k,l)\in\mathbb{Z}^d\times\mathbb{N}_0^d} c_{kl}\psi_{kl} \tag{50}$$

with coefficients which satisfy

$$\|c\|_{\ell^{p,q}_{\tilde{m}}} = \Big(\sum_{l\in\mathcal{N}_0^d}\Big(\sum_{k\in\mathbb{Z}^d}|c_{kl}|^p m(\tfrac{k}{2},l)^p\Big)^{q/p}\Big)^{1/q} < \infty\,. \tag{51}$$

The norm is $\|f\|_{M^{p,q}_m} = \|c\|_{\ell^{p,q}_{\tilde{m}}}$. If $p=q$ we write M^p_m for $M^{p,p}_m$.

If m is a sub-exponential weight and $\ell^{p,q}_m \not\subseteq \ell^2$, then $M^{p,q}_m$ is still defined by (50) and (51) as a subspace of ultradistributions [33, Ch. 11.4].

While it is more convenient for our purpose, Definition 4 is not the standard definition, instead $M_m^{p,q}$ is usually defined by the behavior of the short-time Fourier transform as follows. Fix a "nice" function ψ, e.g., $\psi(t) = e^{-\pi t^2}$ or such that both ψ and $\hat{\psi}$ have exponential decay. The short-time Fourier transform $V_\psi f$ of f with respect to the window ψ is defined by

$$V_\psi f(x,\omega) = \int_{\mathbb{R}} f(t)\overline{\psi(t-x)}e^{-2\pi i\omega\cdot t}\,dt = \langle f, M_\omega T_x \psi\rangle\,. \tag{52}$$

Then $f \in M_m^{p,q}$ if and only if

$$\Big(\int\Big(\int |V_\psi f(x,\omega)|^p m(x,\omega)^p\,dx\Big)^{q/p}\,d\omega\Big)^{1/q} < \infty \tag{53}$$

and this expression is an equivalent norm for $M_m^{p,q}$. See [33, Ch. 12.3] and [26] for the technical details and subtleties.

5.2. **Gabor Frames.** Let $\mathcal{Z}$ be a separated set in the time-frequency plane $\mathbb{R}^{2d}$ and let $g \in L^2(\mathbb{R}^d)$ be a fixed window function. We consider frames of the form $\{\pi(z)g : z \in \mathcal{Z}\}$. If $\mathcal{Z}$ is a lattice in $\mathbb{R}^{2d}$, such a frame is called a *uniform Gabor frame* (or Weyl-Heisenberg frame), whereas for arbitrary sets $\mathcal{Z}$ we speak of *non-uniform Gabor frames*. See [18] or [33, Chs. 5–7,12,13] for the theory of uniform Gabor frames. Non-uniform Gabor frames are not yet fully understood, some results can be found in [8, 12, 32, 34, 41].

A Gabor frame cannot be localized in the strict sense of Definition 2. For simplicity we argue in dimension $d = 1$ and assume that $g \in M^\infty_{v_{s+2+\epsilon}}$, i.e., $|V_\psi g(z)| \le C(1+|z|)^{-s-2-\epsilon}$ according to (53). Then for $l \neq 0$ we estimate that

$$\begin{aligned}
|\langle \pi(z)g, \psi_{kl}\rangle| &= \frac{1}{\sqrt{2}}\left|\langle \pi(z)g, \big(\pi(\tfrac{k}{2}, l) + (-1)^{k+l}\pi(\tfrac{k}{2}, -l)\big)\psi\rangle\right| \\
&\le \frac{1}{\sqrt{2}}\left|\langle g, \pi((\tfrac{k}{2}, l) - z)\psi\rangle\right| + \frac{1}{\sqrt{2}}\left|\langle g, \pi((\tfrac{k}{2}, -l) - z)\psi\rangle\right| \\
&\le C\big(1 + |((\tfrac{k}{2}, l) - z)|\big)^{-s-2-\epsilon} + C\big(1 + |((\tfrac{k}{2}, -l) - z)|\big)^{-s-2-\epsilon}.
\end{aligned}$$

Thus condition (16) is not satisfied when $z \in \mathcal{Z}$ is in the lower half plane. Intuitively this is clear, because the basis functions ψ_{kl} are localized at the symmetric points $(k/2, l)$ and $(k/2, -l)$ in the time-frequency plane. (This was the main insight leading to the construction of orthonormal Wilson bases.)

Although we cannot apply Theorems 3.5 and 3.6 directly, we can overcome this difficulty by looking at the matrix T associated to the frame operator

$$Sf = \sum_{z\in\mathcal{Z}} \langle f, \pi(z)g\rangle \pi(z)g\,.$$

Proposition 5.1. *(a) If $g \in M^\infty_{s+2d+\epsilon}$ for some $\epsilon > 0$, then the matrix entries $T_{(kl),(mn)} = \langle S\psi_{mn}, \psi_{kl}\rangle$ for $(k,l),(m,n) \in \mathbb{Z}^d \times \mathbb{N}_0^d$ satisfy the estimate*

$$|T_{(kl),(mn)}| \le C\Big(1 + |(\frac{k-m}{2}, l-n)|\Big)^{-s-2d-\epsilon} \tag{54}$$

for $(k,l),(m,n) \in \mathbb{Z}^d \times \mathbb{N}_0^d$.

(b) If $|V_\phi g(z)| = \mathcal{O}(e^{-\alpha|z|})$ for some $\alpha > 0$, then the matrix entries $T_{(kl),(mn)}$ satisfy the estimate

$$|T_{(kl),(mn)}| \leq Ce^{-\alpha'|(\frac{k-m}{2},l-n)|} \tag{55}$$

for $(k,l),(m,n) \in \mathbb{Z}^d \times \mathbb{N}_0^d$ and some $\alpha' \in (0,\alpha)$.

Proof. We first express ψ_{kl} as a linear combination of time-frequency shifts [33, p. 271]:

$$\begin{aligned}\psi_{kl}(t) &= \alpha_l \prod_{j=1}^d \big(M_{l_j} + (-1)^{k_j+l_j} M_{-l_j}\big) T_{\frac{k_j}{2}} \psi(t_j) \\ &= \alpha_l \sum_{\eta_j=\pm 1} \prod_{j=1}^d \eta_j^{k_j+l_j}\, M_{\eta_j l_j} T_{\frac{k_j}{2}} \psi(t_j) \\ &= \alpha_l \sum_{\eta\in\{\pm 1\}^d} \eta^{k+l}\, M_{\eta l} T_{\frac{k}{2}} \tilde{\psi}(t) \\ &= \alpha_l \sum_{\eta\in\{\pm 1\}^d} \eta^{k+l}\, \pi(\frac{k}{2},\eta l)\tilde{\psi}(t)\,.\end{aligned} \tag{56}$$

Here $\tilde{\psi}(t) = \prod_{j=1}^d \psi(t_j)$, $0 \leq \alpha_l \leq 1$ is a non-essential normalization factor (its size depends on the number of j with $l_j \neq 0$). Also, $\eta l = (\eta_1 l_1, \dots, \eta_d l_d)$, and the sum runs over all 2^d choices of $\eta \in \{-1,1\}^d$.

Since $|V_\psi g(z)| \leq C(1+|z|)^{-s-2d-\epsilon}$ by the assumption $g \in M^\infty_{v_{s+2d+\epsilon}}$ and since

$$|\langle \pi(w)\psi, \pi(z)g\rangle| = |\langle \psi, \pi(z-w)g\rangle|$$

by [33, Lemma 3.1.3], the inner products $\langle \pi(z)g, \psi_{kl}\rangle$ are estimated by

$$\begin{aligned}|\langle \pi(z)g, \psi_{kl}\rangle| &\leq C \sum_{\eta\in\{\pm 1\}^d} |\langle \pi(z)g, \pi(\frac{k}{2},\eta l)\psi\rangle| \\ &\leq C' \sum_{\eta\in\{\pm 1\}^d} \Big(1 + |(\frac{k}{2},\eta l) - z|\Big)^{-s-2d-\epsilon}.\end{aligned} \tag{57}$$

Therefore the matrix entries of T satisfy the following

$$\begin{aligned}|T_{(kl),(mn)}| &= \Big|\sum_{z\in\mathcal{Z}} \langle \psi_{mn}, \pi(z)g\rangle \overline{\langle \pi(z)g, \psi_{kl}\rangle}\Big| \\ &\leq C \sum_{\eta,\eta'\in\{0,1\}^d} \sum_{z\in\mathcal{Z}} \Big(1+|(\frac{m}{2},\eta n) - z|\Big)^{-s-2d-\epsilon} \Big(1+|(\frac{k}{2},\eta' l) - z|\Big)^{-s-2d-\epsilon}.\end{aligned}$$

Carrying out the sums over $\mathcal{Z}$ and applying Lemma 2.2(a), we arrive at the estimate

$$\begin{aligned}|T_{(kl),(mn)}| &\leq C \sum_{\eta,\eta'\in\{0,1\}^d} \Big(1+|(\frac{k-m}{2},\eta' l - \eta n)|\Big)^{-s-2d-\epsilon} \\ &\leq C'\Big(1+|(\frac{k-m}{2}, l-n)|\Big)^{-s-2d-\epsilon},\end{aligned}$$

since $|\eta n - \eta' l| > |n-l|$ for $\eta \neq \eta'$ and $l, n \in \mathbb{N}_0^d$.

Part (b) is proved similarly by using Lemma 2.2(b). ■

Now the main theorem about nonuniform Gabor frames follows exactly as in the proof of Theorem 3.5.

Theorem 5.2. *Assume that* $\{\pi(z)g : z \in \mathcal{Z}\}$ *is a frame for* $L^2(\mathbb{R}^d)$ *and that* $g \in M^\infty_{v_{s+2d+\epsilon}}$ *for some* $\epsilon > 0$.

(a) Then the frame operator S *is invertible on all spaces* M^p_m *for each* $1 \le p \le \infty$ *and every* s*-moderate weight function* m.

(b) Write the dual frame as $\{\widetilde{e_z} = S^{-1}(\pi(z)g) : z \in \mathcal{Z}\}$. *Then the frame expansions*

$$f = \sum_{z \in \mathcal{Z}} \langle f, \widetilde{e_z}\rangle \pi(z)g = \sum_{z\in\mathcal{Z}} \langle f, \pi(z)g\rangle \widetilde{e_z} \tag{58}$$

converges unconditionally in M^p_m *for* $1 \le p < \infty$ *(and weak-* in* M^∞_m*).*

(c) The modulation space $M^p_m, 1 \le p \le \infty$, *can be characterized by the frame coefficients as follows:*

$$\|f\|_{M^p_m} \asymp \|\langle f, \pi(z)g\rangle\|_{\ell^p_m(\mathcal{Z})} \asymp \|\langle f, \widetilde{e_z}\rangle\|_{\ell^p_m(\mathcal{Z})} . \tag{59}$$

If $|V_\psi g(z)| = \mathcal{O}(e^{-\alpha|z|})$, *then the above conclusions hold for all* M^p_m *with* $1 \le p \le \infty$ *and all sub-exponential weights* m.

Proof. In the proof of Theorem 3.5 we have only used the decay estimates for the entries of T. This point is settled by Prop. 5.1, consequently the proof is now exactly the same as that of Theorem 3.5. ■

REMARKS: 1. For uniform Gabor frames $\{\pi(\alpha k, \beta l)g : k, l \in \mathbb{Z}^d\}$ this theorem was already proved in [33, Thm. 13.5.3]. If the condition $g \in M^\infty_{v_{s+2d+\epsilon}}$ is replaced by the slightly weaker condition $g \in M^1_{v_s}, s \ge 0$, then the dual window $\gamma = S^{-1}g$ is also in $M^1_{v_s}$ [25, 35]. This is quite subtle and uses methods from the theory of symmetric Banach algebras and the representation theory of the Heisenberg group.

2. It can be shown that Theorem 5.2 also holds for the mixed norm modulation spaces $M^{p,q}_m, p \neq q$.

In Theorem 5.2 we have omitted a precise statement about the time-frequency concentration of the dual frame $\tilde{e_z} = S^{-1}(\pi(z)g)$. It is easy to obtain such an estimate from the boundedness of S^{-1} on $M^\infty_{v_s}$, namely,

$$\|\widetilde{e_z}\|_{M^\infty_{v_s}} = \|S^{-1}(\pi(z)g)\|_{M^\infty_{v_s}} \le C\|\pi(z)g\|_{M^\infty_{v_s}} \le C(1+|z|)^s \|g\|_{M^\infty_{v_s}} .$$

This implies that

$$|V_\psi \widetilde{e_z}(w)| \le C(1+|z|)^s\,(1+|w|)^{-s} .$$

We now prove a sharper and uniform decay estimate for the dual frame. In view of (57) we introduce a distance function on the time-frequency plane that reflects the special symmetry of the multivariate Wilson basis. For $w \in \mathbb{R}^{2d}$ and $z = (z_1, \dots, z_{2d}) = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$ with $\zeta_1, \zeta_2 \in \mathbb{R}^d$ we define

$$(60) \quad d(w,z) = \min_{\eta \in \{-1,1\}^d} |w - (\zeta_1, \eta\zeta_2)| = \Big(\sum_{j=1}^{d} (w_j - z_j)^2 + \sum_{j=d+1}^{2d} (|w_j| - |z_j|)^2 \Big)^{1/2} .$$

Note that $|w - z| \geq d(w,z)$ and that the localization property (57) can be recast as

$$|\langle \pi(z)g, \psi_{kl}\rangle| \leq C\Big(1 + d\big((\frac{k}{2}, l), z\big)\Big)^{-s-2d-\epsilon} .$$

Now we can complete the statement of Theorem 5.2 with the natural result about the localization of the dual Gabor frame.

Proposition 5.3. *Assume that $\{\pi(z)g : z \in \mathcal{Z}\}$ is a frame for $L^2(\mathbb{R}^d)$.*

(a) If $g \in M^\infty_{v_{s+2d+\epsilon}}$ for some $\epsilon > 0$, then the dual frame $\widetilde{e_z} = S^{-1}(\pi(z)g)$ satisfies the time-frequency localization estimate

$$|V_\psi \widetilde{e_z}(w)| \leq C\Big(1 + d(w - z)\Big)^{-s-2d-\epsilon}, \qquad z \in \mathcal{Z}, w \in \mathbb{R}^{2d} .$$

(b) If $|V_\psi g(z)| \leq Ce^{-\alpha|z|}$, then for some $\alpha' \in (0, \alpha)$,

$$|V_\psi \widetilde{e_z}(w)| \leq Ce^{-\alpha' d(w,z)}, \qquad z \in \mathcal{Z}, w \in \mathbb{R}^{2d} .$$

Proof. We use a notational short-cut and write $\nu(z) = (1 + |z|)^{-s-2d-\epsilon}$ (or $\nu(z) = e^{-\alpha|z|}$ in case (b)). Furthermore, set $r = (\frac{k}{2}, l)$ and $s = (\frac{m}{2}, n)$ for $k, m \in \mathbb{Z}^d$, $l, n \in \mathbb{N}_0^d$, and write $\eta z = \eta(\zeta_1, \zeta_2) := (\zeta_1, \eta\zeta_2)$ for $z = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$.

To find an estimate for the short-time Fourier transform of $\widetilde{e_z}$ we rewrite it as

$$(61) \qquad \langle \widetilde{e_z}, \pi(w)\psi\rangle = \sum_{r \in \frac{1}{2}\mathbb{Z}^d \times \mathbb{N}_0^d} \langle \widetilde{e_z}, \psi_r\rangle \langle \psi_r, \pi(w)g\rangle .$$

For the second inner product we have already found an estimate in (57). The first inner product is treated as in (25):

$$(62) \qquad \langle \widetilde{e_z}, \psi_r\rangle = \langle S^{-1}(\pi(z)g), \psi_r\rangle = \sum_{s \in \frac{1}{2}\mathbb{Z}^d \times \mathbb{N}_0^d} \langle \pi(z)g, \psi_s\rangle \overline{(T^{-1})_{rs}} .$$

We use Prop. 5.1 for the entries of T^{-1} and (57) for the inner products in these expansions. Substituting these estimates into (61) and (62), we obtain that

$$\begin{aligned} |\langle \widetilde{e_z}, \pi(w)\psi\rangle| &\leq C \sum_{\eta,\eta' \in \{\pm 1\}^d} \sum_{r,s \in \frac{1}{2}\mathbb{Z}^d \times \mathbb{N}_0^d} \nu(\eta s - z)\nu(r - s)\nu(\eta' r - w) \\ &= C \sum_{\eta,\eta' \in \{\pm 1\}^d} \sum_{r,s \in \frac{1}{2}\mathbb{Z}^d \times \mathbb{N}_0^d} \nu(s - \eta z)\nu(r - s)\nu(r - \eta' w) \\ &\leq C' \sum_{\eta,\eta' \in \{\pm 1\}^d} \nu(\eta z - \eta' w) \\ &\leq C' 2^{2d} \nu\big(d(z,w)\big) . \end{aligned}$$

To resolve the sum over r and s we have used Lemma 2.2 twice, and in the last step we have used that $|\eta z - \eta' w| \geq d(z, w)$ for all $\eta, \eta' \in \{-1, 1\}^d$. The last expression is the announced time-frequency localization and we are done. ■

5.3. **Time-Frequency Molecules.** Our final result shows that we can replace the Gabor frame $\{\pi(z)g : z \in \mathcal{Z}\}$ by a frame consisting of *time-frequency molecules* and obtain the same conclusions.

Theorem 5.4. *Let $\mathcal{Z}$ be a separated set in $\mathbb{R}^{2d}$ and let $\{g_z : z \in \mathcal{Z}\}$ be a frame for $L^2(\mathbb{R}^d)$ satisfying the uniform estimates*

$$|V_\psi g_z(w)| \leq C(1 + |w - z|)^{-s-2d-\epsilon}, \qquad z \in \mathcal{Z}. \tag{63}$$

Then the frame operator $Sf = \sum_{z \in \mathcal{Z}} \langle f, g_z \rangle g_z$ is invertible simultaneously on all M^p_m for each $1 \leq p \leq \infty$ and all s-moderate weights m.

The dual frame $\widetilde{g_z} = S^{-1}(g_z)$ satisfies the localization estimates

$$|V_\psi \widetilde{g_z}(w)| \leq C' \big(1 + d(w - z)\big)^{-s-2d-\epsilon} \quad z \in \mathcal{Z}.$$

The frame expansions

$$f = \sum_{z \in \mathcal{Z}} \langle f, g_z \rangle \widetilde{g_z} = \sum_{z \in \mathcal{Z}} \langle f, \widetilde{g_z} \rangle g_z$$

converge unconditionally in the modulation spaces M^p_m for $1 \leq p < \infty$, and

$$A \|f\|_{M^p_m} \leq \Big(\sum_{z \in \mathcal{Z}} |\langle f, \widetilde{\psi_z} \rangle|^p m(z)|^p \Big)^{1/p} \leq B \|f\|_{M^p_m}, \quad f \in M^p_m.$$

The same conclusions hold for exponentially localized time-frequency molecules, i.e., assuming $|V_\psi g_z(w)| \leq C^{-\alpha|w-z|}$, $z \in \mathcal{Z}$.

Proof. In the proof of Theorems 3.5, 5.2, and Prop. 5.3 we have only used the estimates (63), but not the precise form of the frame. Therefore the proof is the same. ■

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