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Analytic Solution of Two Apodization Problems

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Apodization theory is concerned with the determination of the distribution of light over the exit pupil of an optical system required in order to achieve a suppression of the side lobes of the diffraction pattern. Here analytic solutions are given to the problem of determining the distribution of light in the exit pupil to concentrate maximally the illuminance in a geometrically similar region of the image plane. Both slit and circular apertures are treated.

1. INTRODUCTION

APODIZATION theory is concerned with the determination of the distribution of light over the exit pupil of an optical system required in order to achieve a desired distribution of illuminance over a given plane in the image field. Most often the goal is suppression of the side lobes of the diffraction pattern. A comprehensive review of the uses of apodization and of research in this area has recently been given by Jacquinot and Roizen-Dozzier.[1]

Two apodization problems that have received considerable attention are: (a) the determination of that amplitude distribution across a long rectangular (slit) aperture which maximizes the fraction of the total illuminance that lies within a parallel rectangle of given width in the image plane; (b) the determination of that amplitude distribution over a circular pupil which maximizes the fraction of the total illuminance that lies in a prescribed circle in the image plane. This latter problem was considered as early as 1935 by Straubel.[2] An integral-equation formulation was given by Luneberg[3] in 1944. More recently, Lansraux and Boivin[4] discussed problem (b) in considerable detail and, by making use of polynomial expansions of the pupil amplitude distribution, were able to present extensive approximate numerical results. Barakat,[5] using different approximation techniques, gave numerical data for both (a) and (b). Additional references to earlier work on these apodization problems can be found in the papers and the review article[1] already cited.

In the present paper, we present complete analytic solutions to both problems and provide some numerical detail. Our task is largely one of transcribing results from other disciplines into optical terms, for the mathematical problems behind (a) and (b) have received extensive treatment elsewhere. We quote freely from this nonoptical literature as needed.

[1] P. Jacquinot and B. Roizen-Dossier in *Progress in Optics*, E. Wolf, ed. (North-Holland Publishing Co., Amsterdam, 1964), Vol. III, p. 31.

[2] R. Straubel, *Pieter Zeeman. Verhandelingen op 25 Mei 1935 Aangeboden aan Prof. Dr. P. Zeeman* (Martinus Nijhoff, The Hague, Netherlands, 1935), p. 302.

[3] R. K. Luneberg, *Mathematical Theory of Optics* (University of California Press, Berkeley, California, 1964), p. 353.

[4] G. Lansraux and G. Boivin, Can. J. Phys. 39, 158 (1961).

[5] R. Barakat, J. Opt. Soc. Am. 52, 264 (1962).

2. MATHEMATICAL PROBLEM

Let $\mathbf{x}'=(x_1',x_2')$ be the radius vector in the plane of the exit pupil from the optical axis to an arbitrary point in that plane; let $\boldsymbol{\xi}'$ be the radius vector in the image plane from the optical axis to a point in the image. Then, under the usual physical assumptions,[1] the light amplitude $A'(\boldsymbol{\xi}')$ in the image plane is proportional to

$$\int_{|\mathbf{x}'|\leq a} e^{i(k/p)\boldsymbol{\xi}'\cdot\mathbf{x}'}T'(\mathbf{x}')dx_1'dx_2',$$

where T' is the light amplitude in the circular exit pupil of radius a. Here k is the wavenumber (2π/wavelength) of the light and p is the distance from the pupil to the image plane. Following Jacquinot and Roizen-Dossier,[1] we introduce normalized coordinates and amplitude functions

$$\boldsymbol{\xi}=(ak/\pi p)\boldsymbol{\xi}',\qquad \mathbf{x}=(1/2a)\mathbf{x}',\qquad A(\boldsymbol{\xi})=A'(\boldsymbol{\xi}'),\qquad T(\mathbf{x})=T'(\mathbf{x}'), \tag{1}$$

and write

$$A(\boldsymbol{\xi})=\iint_{|\mathbf{x}|\leq\frac{1}{2}} e^{2\pi i\boldsymbol{\xi}\cdot\mathbf{x}}T(\mathbf{x})dx_1dx_2. \tag{2}$$

Apodization problem (b) then requires finding the function $T(\mathbf{x})$ for which the ratio

$$\lambda=\iint_{|\boldsymbol{\xi}|\leq\rho}|A(\boldsymbol{\xi})|^2d\xi_1d\xi_2\Big/\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|A(\boldsymbol{\xi})|^2d\xi_1d\xi_2 \tag{3}$$

is a maximum. Here

$$\rho=(ak/\pi p)b \tag{4}$$

is a normalized measure of the radius b of the circle in which the illuminance is to be maximally concentrated. The circle is centered on the optical axis.

This problem differs only in notation from a special case of a more general problem treated in a recent publication[6] by the author. To facilitate adaptation to the apodization problems of results obtained there, we establish the notation of that paper. Let points in

[6] D. Slepian, Bell System Tech. J. 43, 3009 (1964).

Euclidean space of D dimensions E_D be denoted by vectors $\mathbf{x}=(x_1,x_2,\cdots,x_D)$. Let two square-integrable functions f and F be related by

$$f(\mathbf{x})=(2\pi)^{-D}\int_R e^{i\mathbf{x}\cdot\mathbf{y}}F(\mathbf{y})d\mathbf{y}, \tag{5}$$

where R is the unit sphere $\sum y_i^2\leq 1$, $\mathbf{x}\cdot\mathbf{y}=\sum x_iy_i$ is the usual scalar product and we write dy for Πdy_i. What is the maximum value of

$$\lambda=\int_C |f(x)|^2dx\int\int_{E_D}|f(x)|^2dx \tag{6}$$

for all f of the form (5)? Here C is the sphere $|\mathbf{x}|\leq c$ of radius $c>0$.

It is easy to show[6] that this maximum is λ_0, the largest eigenvalue of the integral equation

$$\lambda\psi(\mathbf{x})=\int_R K_D(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})d\mathbf{y} \tag{7}$$

with kernel

$$K_D(\mathbf{x}-\mathbf{y})=\left(\frac{c}{2\pi}\right)^D\int_R e^{ic\mathbf{z}\cdot(\mathbf{x}-\mathbf{y})}d\mathbf{z}. \tag{8}$$

This largest eigenvalue is nondegenerate. Denote the corresponding eigenfunction by $\psi_0(\mathbf{x})$. Then the function f (unique up to a constant factor) of form (5) for which (6) achieves the maximum value λ_0 is given by (5) with $F(\mathbf{y})=\psi_0(\mathbf{y})$. ψ_0 can be chosen to be real.

Equation (8) suggests consideration of the simpler "square root" equation

$$\alpha\psi(\mathbf{x})=\int_R e^{ic\mathbf{x}\cdot\mathbf{y}}\psi(\mathbf{y})d\mathbf{y}. \tag{9}$$

The complete sets of eigenfunctions of (7) and (9) are indeed identical[6] and

$$\lambda=(c/2\pi)^D|\alpha|^2. \tag{10}$$

Note that K_D, and hence λ, ψ, and α all depend on c. We have suppressed this dependence in our notation. All quantities, of course, depend also on D.

The solutions of (9), initially defined only for $\mathbf{x}\epsilon R$, can be extended by the right-hand side of (9) to all of E_D. These extended functions can be normalized to possess the interesting double-orthogonality property

$$\int_R \psi_m(\mathbf{x})\psi_n(\mathbf{x})dx=\delta_{mn}=\lambda_n\int_{E_D}\psi_m(\mathbf{x})\psi_n(\mathbf{x})dx. \tag{11}$$

Other properties of the solutions are given in later sections. Here we note that with $D=2$, the apodization problem (b) is solved by choosing the normalized pupil amplitude density

$$T(\mathbf{x})=\psi_0(2\mathbf{x}) \tag{12}$$

with

$$c=kab/p=\pi\rho. \tag{13}$$

Comparison of (9) and (2) shows that the corresponding optimal, normalized, light amplitude distribution in the image plane is

$$A(\xi)=(\alpha/2^D)\psi_0[(\pi/c)\xi]. \tag{14}$$

Precisely the same formalism with $D=1$ gives solution of the slit apodization problem (a). Here the vector ξ' of (1) measures distance across the slit of total width $2a$, and $\mathbf{x}'$ measures distance along a parallel direction in the image plane. The illuminance is maximized in the plane within a rectangle of width $2b$ centered on the optical axis and parallel to the slit pupil.

Following Jacquinot and Roizen-Dossier,[1] we normalize the solution so that $T(0)=\psi_0(0)=1$. Their measures of apodization[7] can now be written

$$\epsilon(\rho)=\lambda_0, \tag{15}$$

and for $D=1$

$$\frac{I(0)}{I_0(0)}=\frac{\alpha_0^2}{4}=\frac{\pi\lambda_0}{2c},\quad \frac{\tau}{\tau_0}=\tfrac{1}{2}N, \tag{16}$$

while for $D=2$

$$\frac{I(0)}{I_0(0)}=\frac{\alpha_0^2}{\pi^2}=\frac{4\lambda_0}{c^2},\quad \frac{\tau}{\tau_0}=\frac{1}{\pi}N, \tag{17}$$

with

$$N=\int_{|x|\leq 1}\psi_0^2(\mathbf{x})dx. \tag{18}$$

Here ϵ, the "encircled energy factor," is the fraction of the energy in the diffraction pattern lying within the region $|\xi'|\leq b$. The quantity $I(0)/I_0(0)$ is the ratio of the intensity at the center of the apodized diffraction pattern to the intensity at the center of the pattern that would result with uniform illumination without apodization ($T=1$). The quantity τ/τ_0 is the ratio of the energy in the apodized diffraction pattern to that obtained without apodization.

3. SLIT APERTURE. $D=1$

When $D=1$, Eqs. (7) and (8) become

$$\lambda\psi(x)=\int_{-1}^{1}\frac{\sin c(x-y)}{\pi(x-y)}\psi(y)dy. \tag{19}$$

This equation has applications in many fields and has been studied intensively in recent years by workers in communication theory.[8,9] In this connection it was noted at least as early as 1954[10] that its solutions are the prolate spheroidal wavefunctions[11–13] of zero order.

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[7] P. Jacquinot and B. Roizen-Dossier in *Progress in Optics III* edited by E. Wolf (North Holland Publishing Co., Amsterdam, 1964), pp. 47, 78.

[8] D. Slepian and H. O. Pollak, Bell System Tech. J. **40**, 43 (1961).

[9] H. J. Landau and H. O. Pollak, Bell System Tech. J. **40**, 65 (1961); **41**, 1295 (1962).

Applications to lasers were given by Boyd and Gordon in 1961.[14] Details concerning the behavior of the λ and ψ for large c have been given recently by the author.[15] We quote freely from this literature, translating by means of (12), (13), (14) where necessary. Superscripts just before equations indicate footnotes with references giving more detail about the equations.

The eigenfunctions of (19) are also the continuous solutions of [8,12]

$$\frac{d}{dx}(1-x^2)\frac{d\psi}{dx}+(\chi-c^2x^2)\psi=0 \tag{20}$$

that remain bounded at $x=\pm 1$. The solution ψ_0 of this equation belonging to the smallest eigenvalue χ_0 is the eigenfunction of (19) belonging to the largest value of λ.[8]

From (20), various series expansions for $\psi_0(x)$ can be constructed. If c is not too large (say $c\leq 10$), ψ_0 can be conveniently computed from expansions in either Legendre polynomials $P_r(x)$,[12]

$$\psi_0(x)=\sum_{r=0}^{\infty} d_r(c)P_r(x) \tag{21}$$

for $|x|<1$, or in terms of spherical Bessel functions[12]

$$\psi_0(x)=K\sum_{r=0}^{\infty} d_r(c)j_r(cx) \tag{22}$$

valid for all x. The d's satisfy a three-term recurrence and can be calculated with great accuracy using the method of Bouwkamp, as explained for example in Flammer.[12] In terms of these coefficients,[12]

$$\lambda_0=\frac{2c}{\pi}d_0^2\Big/\left[\sum_{j=0}^{\infty}\frac{(-1)^j}{2^{2j}}\binom{2j}{j}d_{2j}\right]^2, \tag{23}$$

$$N=2\sum_{j=0}^{\infty}\frac{d_{2j}^2}{4j+1}. \tag{24}$$

Tables of the d's for small values of c are available.[12,13]

For very small values of c,[15]

$$\lambda_0=(2/\pi)c[1-(c^2/9)+0(c^4)], \tag{25}$$

while for very large c,[15]

$$1-\lambda_0=4(\pi c)^{1/2}e^{-2c}[1-(3/32c)+0(c^{-2})]. \tag{26}$$

The lead term of this expansion was first found by Fuchs.[16]

The eigenfunction $\psi_0(x)$ is even in x. When normalized so that $\psi_0(0)=1$, it approaches unity for each x as c approaches zero. For $|x|\leq 1$, we have for small c[15]

$$\psi_0(x)=1+(c^2/18)(1-3x^2)+0(c^4). \tag{27}$$

For large c, the behavior is more complicated[15]:

$$\psi_0(x)\sim\begin{cases} e^{-cx^2/2}, & 0\leq x\leq c^{-1/2} \\ \dfrac{\sqrt{2}e^{-c}e^{c(1-x^2)^{1/2}}}{(1-x^2)^{1/4}[1+(1-x^2)^{1/2}]^{1/2}}, & c^{-1/2}\leq x\leq 1-c^{-1} \\ 2(c\pi)^{1/2}e^{-c}I_0[c(1-x^2)^{1/2}], & 1-c^{-1}\leq x\leq 1 \\ 2(c\pi)^{1/2}e^{-c}J_0[c(x^2-1)^{1/2}], & 1\leq x\leq 1+(1/c) \\ \dfrac{2^{3/2}e^{-c}\cos[c(x^2-1)^{1/2}-\frac{1}{2}\arctan(x^2-1)^{1/2}-\pi/4]}{x^{1/2}(x^2-1)^{1/4}}, & 1+(1/c)\leq x \end{cases} \tag{28}$$

where J_0 is the usual Bessel function and $I_0(u)=J_0(iu)$ is the modified Bessel function. Higher-order terms in this asymptotic expansion are available.[15]

For the normalization constant N of (18), we have[15] $N=2+0(c^4)$ for small c and $N\sim(\pi/c)^{1/2}[1+0(c^{-2})]$ for large c. For the measures (15)–(16) of apodization, then, we have for small ρ

$$\begin{aligned}\epsilon(\rho)&=2\rho[1-(\pi^2/9)\rho^2+0(\rho^4)],\\ I(0)/I_0(0)&=1-(\pi^2/9)\rho^2+0(\rho^4),\\ \tau/\tau_0&=1+0(\rho^4),\end{aligned} \tag{29}$$

and for large ρ

$$\begin{aligned}\epsilon(\rho)&\sim 1-4\pi\rho^{1/2}e^{-2\pi\rho}[1+0(\rho^{-1})],\\ I(0)/I_0(0)&\sim(1/2\rho)-(2\pi e^{-2\pi\rho}/\rho^{1/2})[1+0(\rho^{-1})],\\ \tau/\tau_0&\sim(1/2\rho^{1/2})[1+0(\rho^{-2})].\end{aligned} \tag{30}$$

4. THE CIRCULAR APERTURE. $D=2$

Equation (9) with $D=2$ has applications in the theory of confocal lasers.[17–19] Its solutions may be useful in optical applications other than apodization problem (b), for, as $c\to 0$, the radial component of the eigenfunctions reduces to the Zernike polynomials[20] of use in the diffraction theory of aberrations.

[10] D. Slepian, IRE Trans. **PGIT-3**, 68 (1954).
[11] J. Meixner and F. W. Schäfke, *Mathieusche Funktionen und Sphäroidfunktionen* (Springer-Verlag, Berlin, 1954).
[12] C. Flammer, *Spheroidal Wave Functions* (Stanford University Press, Stanford, California, 1957).
[13] J. A. Stratton, P. M. Morse, L. J. Chu, J. D. C. Little, and P. J. Corbató, *Spheroidal Wave Functions* (John Wiley & Sons, Inc., New York, 1956).
[14] G. D. Boyd and J. P. Gordon, Bell System Tech. J. **40**, 489 (1961).
[15] D. Slepian, J. Math. Phys. (MIT) **44**, 99 (1965).
[16] W. Fuchs, J. Math. Anal. Appl. **9**, 317 (1964).
[17] A. G. Fox and Tingye Li, Bell System Tech. J. **40**, 453 (1961).
[18] J. C. Heurtley in *Proc. Symposium on Quasi-Optics* (Polytechnic Press, Brooklyn, New York, 1964), p. 367.
[19] H. Kogelnik, in *Advances in Lasers*, A. K. Levine, ed. (Dekker Publishers, New York, 1965).
[20] M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, London, 1959).

With $D=2$, the eigenfunctions of (7) and (9) can be written in the form[6]

$$\begin{aligned} \psi_{0,n}(x_1,x_2)&=R_{0,n}(r), & \alpha_{0,n}&=2\pi\beta_{0,n},\\ \psi_{N,n}(x_1,x_2)&=R_{N,n}(r)\begin{matrix}\cos N\theta\\ \sin N\theta,\end{matrix} & \alpha_{N,n}&=2\pi\beta_{N,n}, \\ N&=1,2,\cdots, \quad n=0,1,2,\cdots, \end{aligned} \tag{31}$$

where $r=(x_1^2+x_2^2)^{\frac{1}{2}}$ and $\theta=\arctan x_2/x_1$ are polar coordinates and[6]

$$\beta_{N,n}R_{N,n}(r)=\int_0^1 J_N(crr')R_{N,n}(r')r'dr', \qquad n,N=0,1,2,\cdots. \tag{32}$$

This integral equation which determines the radial factor of $\psi(\mathbf{x})$ can be written in the more symmetric form[6]

$$\gamma_{N,n}\varphi_{N,n}(r)=\int_0^1 J_N(crr')(crr')^{\frac{1}{2}}\varphi_{N,n}(r')dr', \tag{33}$$

where[6]

$$\gamma_{N,n}=c^{\frac{1}{2}}\beta_{N,n}, \qquad \varphi_{N,n}(r)=r^{\frac{1}{2}}R_{N,n}(r). \tag{34}$$

The functions $\varphi_{N,n}(r)$ just introduced are called[6] generalized prolate spheroidal functions. They are the continuous solutions of the differential equation[6,18]

$$\frac{d}{dr}(1-r^2)\frac{d\varphi}{dr}+\left(\frac{\frac{1}{4}-N^2}{r^2}-c^2r^2+\chi\right)\varphi=0 \tag{35}$$

that remain bounded at $r=0$ and $r=1$. Such solutions exist only for certain discrete values of χ. The smallest eigenvalue, say $\chi_{0,0}$, occurs when $N=0$ in (35). The corresponding eigenfunction, $\varphi_{0,0}(r)$, gives rise through (31) and (34) to the eigenfunction $\psi_{0,0}(x_1,x_2)=\varphi_{0,0}(r)/r^{\frac{1}{2}}$ of (7), for $D=2$ having the largest value of λ.

If c is not too large (say $c\leq 10$), $\varphi_{N,n}(r)$ can be conveniently computed for $0\leq r\leq 1$ by an expansion of the form[6]

$$\varphi_{N,n}(r)=\sum_{j=0}^{\infty} d_j^{N,n}(c)T_{N,j}(r), \tag{36}$$

where[6]

$$T_{N,n}(r)=\binom{n+N}{n}^{-1} r^{N+\frac{1}{2}}P_n^{(N,0)}(1-2r^2) \tag{37}$$

and $P_n^{(\alpha,\beta)}(x)$ is a Jacobi polynomial.[21] An alternative expansion valid for all $r\geq 0$ is[6]

$$\varphi_{N,n}(r)=\frac{1}{\gamma_{N,n}}\sum_{j=0}^{\infty} d_j^{N,n}(c)\frac{J_{N+2j+1}(cr)}{\binom{N+j}{j}(cr)^{\frac{1}{2}}}. \tag{38}$$

The $d_j^{N,n}$ satisfy a three-term recurrence and can be calculated with great accuracy using the method of Bouwkamp.[12] In terms of these d's,[6]

$$\alpha_{0,0}=\pi d_0^{00}/\sum_j d_j^{00} \tag{39}$$

and from (10)

$$\lambda_{0,0}=(c/2\pi)^2\alpha_{0,0}^2. \tag{40}$$

The normalization constant N of (18) is given by

$$N=\pi\sum_{j=0}^{\infty}\frac{(d_j^{0,0})^2}{2j+1}\Big/\left[\sum_{j=0}^{\infty} d_j^{00}(c)\right]. \tag{41}$$

For small values of c,[6]

$$\begin{aligned} \alpha_{0,0}&=\pi[1-(c^2/16)+0(c^4)],\\ \lambda_{0,0}&=(c^2/4)[1-(c^2/8)+0(c^4)],\\ N&=\pi[1-(c^2/8)+0(c^4)], \end{aligned} \tag{42}$$

while for large values of this parameter[6]

$$\begin{aligned} \alpha_{0,0}&\sim(2\pi/c)\{1-4\pi ce^{-2c}[1+0(c^{-1})]\},\\ 1-\lambda_{0,0}&\sim 8\pi ce^{-2c}[1+0(c^{-1})],\\ N&\sim(\pi/c^{\frac{1}{2}})[1+0(c^{-1})]. \end{aligned} \tag{43}$$

For the eigenfunction of interest to us here, we have for small c and $0\leq r\leq 1$[6]

$$\psi_{0,0}=1-(c^2r^2/8)+0(c^4). \tag{44}$$

For large c, the behavior is more complicated[15]:

$$\psi_{0,0}(r)\sim\begin{cases} e^{-(cr^2/2)}, & 0\leq r\leq c^{-\frac{1}{2}}\\ \dfrac{2e^{-c}e^{c(1-r^2)^{\frac{1}{2}}}}{(1-r^2)^{\frac{1}{4}}[1+(1-r^2)^{\frac{1}{2}}]} & c^{-\frac{1}{2}}\leq r\leq 1-c^{-1}\\ 2^{\frac{1}{2}}(\pi c)^{\frac{1}{2}}e^{-c}I_0[c(1-r^2)^{\frac{1}{2}}], & 1-c^{-1}\leq r\leq 1\\ 2^{\frac{1}{2}}(\pi c)^{\frac{1}{2}}e^{-c}J_0[c(r^2-1)^{\frac{1}{2}}], & 1\leq r\leq 1+c^{-1}\\ \dfrac{4e^{-c}\cos[c(r^2-1)^{\frac{1}{2}}-\arctan(r^2-1)^{\frac{1}{2}}-\pi/4]}{(r^2-1)^{\frac{1}{4}}r}, & 1+c^{-1}\leq r. \end{cases} \tag{45}$$

This formula shows that the asymptotic approximation proposed by Lansraux and Boivin,[4] namely, $\psi_{0,0}(r)\sim e^{-cr^2/2}$, $0\leq r\leq 1$, is accurate near the optical axis. For large values of c, however, it becomes inaccurate at the edge of the pupil. For their approximation, we have $\psi_{0,0}(1)/\psi_{0,0}(0)\sim e^{-c/2}$, while from (31) the true asymptotic behavior is $\psi_{0,0}(1)/\psi_{0,0}(0)\sim 2^{\frac{1}{2}}(\pi c)^{\frac{1}{2}}e^{-c}$. Likewise, the illuminance of the diffraction pattern at the edge of the circle of maximum illuminance is much smaller than that given by their approximation.

Equations (42) and (43) give for the measures (15) and (17) of apodization

$$\begin{aligned} \epsilon(\rho)&=(\pi^2\rho^2/4)[1+0(\rho^4)],\\ I(0)/I_0&=1-(\pi^2\rho^2/8)+0(\rho^4),\\ \tau/\tau_0&=1-(\pi^2\rho^2/8)+0(\rho^4), \end{aligned} \tag{46}$$

[21] G. Szegö, *Orthogonal Polynomials* (Am. Math. Soc. Colloquium Publications, Vol. XXIII, New York, 1959), Chap. V.

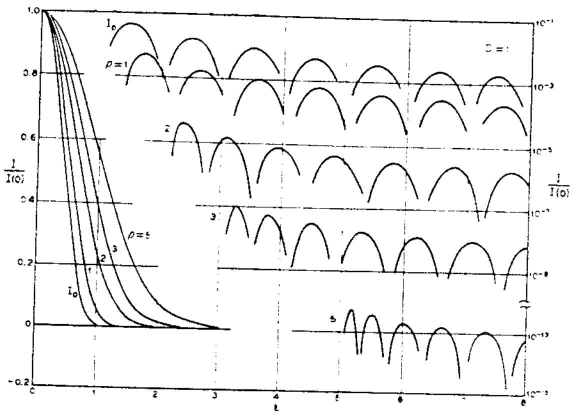

FIG. 1. Relative illuminance of diffraction pattern for apodized slit aperture.

valid for small ρ, while for large ρ we find

$$\epsilon(\rho)=1-8\pi^2\rho e^{-2\pi\rho}[1+O(\rho^{-1})],$$
$$I(0)/I_0=(4/\pi^2\rho^2)-(32/\rho)e^{-2\pi\rho}[1+O(\rho^{-1})], \quad (47)$$
$$\tau/\tau_0=(1/\pi\rho)^{\frac{1}{2}}[1+O(\rho^{-1})].$$

5. NUMERICAL RESULTS

The illuminance (ξ) in the diffraction pattern is proportional to $|A(\xi)|^2$. On Fig. 1 the relative intensity $I(\xi)/I(0)$ is shown for the slit aperture for four different degrees of apodization corresponding to $\rho=1$, 2, 3, and 5. The curve labelled I_0 is a plot of $(\sin\pi\xi)^2/(\pi\xi)^2$, the relative illuminance in the diffraction pattern for a uniformly illuminated slit ($\rho=0$). The plot shows the relative illuminance of the central peak of the diffraction pattern plotted on a linear scale given at the left margin, while the side lobes, or lines, are plotted on a logarithmic scale given at the right margin.

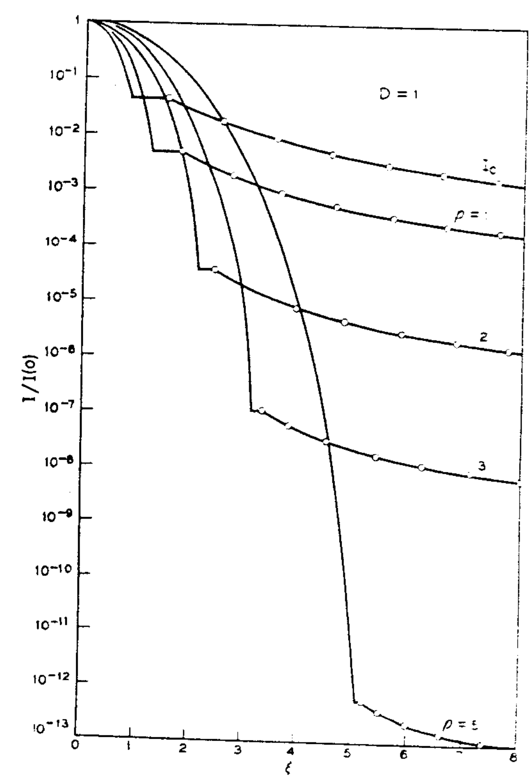

FIG. 2. Upper bound on relative illuminance of diffraction pattern for apodized slit aperture.

TABLE I. Measures of apodization for the slit aperture.

ρ	$\epsilon(\rho)$	$I(0)/I_0(0)$	τ/τ_0
0.5	0.783369	0.783369	0.793310
1.0	0.981046	0.490523	0.548608
1.5	0.998892	0.332964	0.430275
2.0	0.999943	0.249986	0.366433
2.5	0.999997	0.199999	0.324985
3.0	1.00000	0.166667	0.295143
5.0	1.00000	0.100000	0.226460

The linear scale does not exhibit clearly the long, very low tails of the center lobe of the diffraction pattern for the larger values of ρ. Figure 2 shows the relative illuminance of this central lobe on a logarithmic scale. The more nearly horizontal section of each curve is an envelope passed through the maxima of the side lobes and extended horizontally from the first side lobe until it intersects the main lobe. Dots indicate the positions of these maxima. The curve labelled $\rho=2$, for example, gives an upper bound to the relative illuminance to be found in the diffraction pattern when the slit is apodized to concentrate the pattern within a rectangle of half-width $\rho=2$. The bound is achieved all along the rapidly falling portions of the curve and by the side lobes at the dots on the slowly falling curve.

Table I gives values of the measures of apodization (15)–(18) for a number of ρ values.

There are relatively few results in the literature to compare with the numbers found here. Those given by Barakat[9] for $T(x)$ agree rather well with ours for $\rho=1$ near $x=0$, but at the edge of the aperture ($x=\frac{1}{2}$) they are 5% larger. The three-term Fourier-series approximations reported by Jacquinot and Roizen-Dossier[1] for the diffraction pattern do not agree well

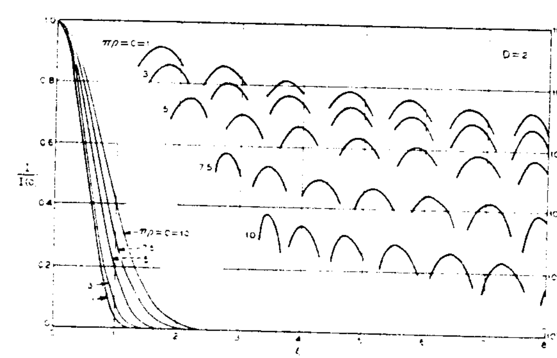

FIG. 3. Relative illuminance of diffraction pattern for apodized circular aperture.

TABLE II. Measures of apodization for the circular aperture.

$c=\pi\rho$	$\epsilon(\rho)$	$I(0)/I_0(0)$	τ/τ_0
1.0	0.221115	0.884460	0.885609
2.0	0.629630	0.629630	0.642386
3.0	0.887050	0.394245	0.430809
4.0	0.974951	0.243738	0.301964
5.0	0.995342	0.159255	0.229356
7.5	0.999949	0.0711075	0.144308
10.0	1.000000	0.0400000	0.105787

with our results. The first side lobes frequently disagree by factors of 10 or more.

Figures 3 and 4 and Table II are similar to Figures 1 and 2 and Table I but treat the case of a circular aperture ($D=2$). Curves are shown for $\rho=1/\pi$, $3/\pi$, $5/\pi$, $7.5/\pi$, and $10/\pi$. The illuminance for uniform illumination of the aperture $4J_1^2(\pi\xi)/(\pi\xi)^2$ lies too close to that for $\rho=1/\pi$ to be shown on the figures.

For the cases $\rho=2/\pi$, $3/\pi$, and $5/\pi$, numerical agreement for λ_0 and $T(x)$ was obtained with the results of Lansraux and Boivin[4] to all significant figures reported by these authors.

6. DISCUSSION

Perhaps the most interesting results of the present investigation are the asymptotic forms (28), (30), (45), and (47). Even for relatively small ρ, say $\rho\geq 2$, they provide a reasonably accurate numerical portrayal of the diffraction pattern. Their analytic form permits a better qualitative understanding of the results of apodization as well.

For the slit aperture, it is seen from (13), (14), and (28) that for large ρ the half-width of the central lobe of the apodized diffraction pattern is proportional to $\rho^{\frac{1}{2}}$; i.e., $I(\xi)/I(0)$ has the value e^{-a_0} for $\xi=(\rho a_0/\pi)^{\frac{1}{2}}$. The first zero of the diffraction pattern does not occur until $\xi\approx\rho[1+\frac{1}{2}(\alpha/\pi\rho)^2]$, where $\alpha=2.4048$ is the first zero of $J_0(x)$. The central lobe has long small tails whose length becomes a larger fraction of the lobe as ρ is increased. At $\xi=\rho$, the relative intensity has been reduced to $I(\rho)/I(0)=4\pi^2\rho e^{-2\pi\rho}$, and beyond the first few fringes the pattern remains small and falls off slowly with ξ as $I(\xi)/I(0)\sim 2\rho[I(\rho)/I(0)](\sin\pi\xi)^2/(\pi\xi)^2$. Roughly, then, apodization by an amount ρ increases the width of the central lobe in proportion to the square root of ρ: The illuminance outside the central lobe is suppressed by a factor *exponentially* small in ρ.

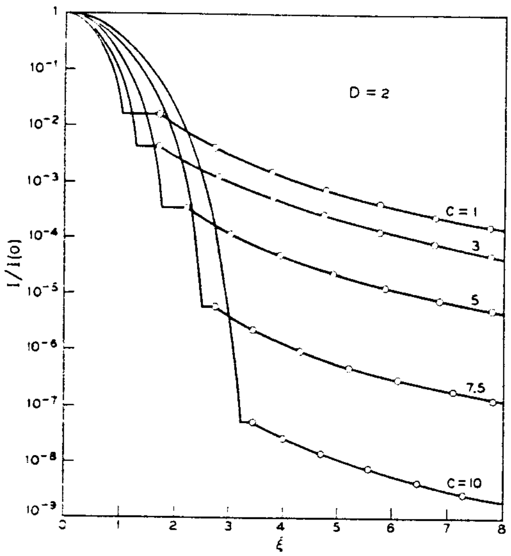

FIG. 4. Upper bound on relative illuminance of diffraction pattern for apodized circular aperture.

The general behavior of the diffraction pattern for the apodized circular aperture is similar. Details can be had from (45), (13), and (14). The half-width of the central lobe and the first zero of the diffraction pattern are as before. We now find $I(\rho)/I(0)=8\pi^2\rho e^{-2\pi\rho}$ and for the outer rings $I(\xi)/I(0)\sim 4\pi\rho^2[I(\rho)/I(0)](\sin\pi\xi)^2/(\pi\xi)^3$.

The asymptotic forms (30) and (47) allow assessment of the many compromises that must be made to achieve this extreme suppression of side lobes.

John Sanderson, NRL, and Archie Mahan, OSA Treasurer, at Dallas meeting.