MAT 280: Applied & Computational Harmonic Analysis
Supplementary Notes III by Naoki Saito

A Brief History of the Convergence of the Fourier Series

Theorem 1 (Dirichlet, 1829) Suppose f is 1-periodic, piecewise smooth on $\mathbb{R}$. Then, nth partial sum, $S_n[f](x) \triangleq \sum_{-n}^{n} c_k \mathrm{e}^{2\pi \mathrm{i} k x}$, satisfies

$$\lim_{n\to\infty} S_n[f](x) = \frac{1}{2}\left[f(x+) + f(x-)\right].$$

In particular, if x is a point of continuity, then $\lim_{n\to\infty} S_n[f](x) = f(x)$.

Theorem 2 (du Bois Reymond, 1876) There exists $f \in C(I)$ such that $\{S_n[f](0)\}$ diverges, where I is an interval of unit length.

Theorem 3 (A weak version of Fejér's Theorem) If f is 1-periodic, *continuous*, and piecewise smooth on $\mathbb{R}$, then the Fourier series of f converges to f *absolutely* and *uniformly*.

Definition: Suppose a series of functions $\sum_1^\infty g_n(x)$ converges to $g(x)$ on a set $x \in I$. Then, the convergence is called *absolute* if $\sum_1^\infty |g_n(x)|$ also converges for $x \in I$.
If we have $\sup_{x\in I} \left| g(x) - \sum_1^N g_n(x) \right| \to 0$ as $N \to \infty$, then we call this a *uniform* convergence.

Theorem 4 (Fejér 1904) If $f \in C(I)$, then the Cesàro means of $S_n[f]$ converge *uniformly* to f.

Definition: The mth *Cesàro mean* of partial sums is the mean of the first $m+1$ partial sums, i.e., $\sigma_m[f](x) \triangleq \frac{1}{m+1}\sum_{n=0}^{m} S_n[f](x)$.

Theorem 5 (Size of the Fourier coeficients and the smoothness of the functions) Suppose f is 1-periodic. If $f \in C^{k-1}(\mathbb{R})$ and $f^{(k-1)}$ is piecewise smooth (i.e., $f^{(k)}$ exists and piecewise continuous), then the Fourier coefficients of f, c_n, satisfy $\sum_n |n^k c_n| < \infty$. In particular, $n^k c_n \to 0$. On the other hand, suppose $c_n, n \neq 0$, satisfy $|c_n| \leq C|n|^{-(k+\gamma)}$ for some $C > 0$ and $\gamma > 1$. Then $f \in C^k(\mathbb{R})$.

Theorem 6 (Kolmogorov, 1926) There exists $f \in L^1(I)$ such that $\{S_n[f](x)\}$ diverges for every x.

Theorem 7 (Carleson, 1966) If $f \in L^2(I)$, then $S_n[f](x)$ converges to $f(x)$ almost everywhere.

Theorem 8 (Hunt, 1967) If $f \in L^p(I), p > 1$, then $S_n[f](x)$ converges to $f(x)$ almost everywhere.

Mathematicians are still trying to simplify the proof of the Carlson-Hunt theorem as of today.

For the details of the above facts, see [1, Chap. 1,2], [2, Chap. 1], [3, Part 1], and [4, Chap. 1]. [5, Chap. 1].

References

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