MAT 280: Applied & Computational Harmonic Analysis
Supplementary Notes II by Naoki Saito

The Generalized Functions

- The generalized functions have more singular behavior than functions (thus the name "generalized functions"), and are always defined as *linear functionals* on the *dual space*. Thus, before we discuss the generalized functions, we need to know the following.

  Definition: Let $\mathcal{X}$ be a vector space over, say, $\mathbb{C}$. A linear map from $\mathcal{X}$ to $\mathbb{C}$ is called a *linear functional* on $\mathcal{X}$. If $\mathcal{X}$ is a normed vector space, then the space $\mathcal{L}(\mathcal{X}, \mathbb{C})$ of *bounded* linear functionals on $\mathcal{X}$ is called the *dual space*, and denoted by $\mathcal{X}^*$ (or $\mathcal{X}'$).
  Examples: The dual of $L^p(\mathbb{R})$, $1 < p < \infty$, is $L^q(\mathbb{R})$, where $(1/p) + (1/q) = 1$. These numbers are called *conjugate exponents*. In particular, L^2 is self dual. Similarly, the dual of the sequence space $\ell^p(\mathbb{Z})$ is $\ell^q(\mathbb{Z})$.

  Hölder's Inequality: Let p and q are conjugate exponents. Then for any $f \in L^p$, $g \in L^q$, we have

  $$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

  (As you can see, the Cauchy-Schwarz inequality is a special version of this with $p = q = 1/2$. The proof is a great exercise.)

  The Riesz Representation Theorem: Suppose p and q are conjugate exponents with $1 < p < \infty$. Then for each linear functional $\varphi \in (L^p)^*$, there exists $g \in L^q$ such that $\varphi(f) = \int f(x) g(x) \, \mathrm{d}x$ for all $f \in L^p$. In other words, $(L^p)^*$ is isometrically isomorphic to L^q.

- The more singular the class of the generalized functions, the more regular its dual.

- We now define the *Schwartz class* $\mathcal{S} = \{f \in C^\infty(\mathbb{R}) \mid \sup_{x \in \mathbb{R}} |x^k \partial^\ell f| < \infty, \text{for any } k, \ell \in \mathbb{N}\}$, which are very smooth and decay faster than any polynomial at infinity, i.e., a very nice class of functions. An example: The Gaussian $g(x) = \mathrm{e}^{-x^2}$.

- Then, we consider the dual $\mathcal{S}'$. You can imagine that members of this class can be very singular or "spiky." This dual space is called the *tempered distributions*. Being as a linear functional, each member of $\mathcal{S}'$ acts on the Schwartz functions. More precisely, if $F \in \mathcal{S}'$ and $\phi \in \mathcal{S}$, then the value of F at ϕ (F is a *linear map* from $\mathcal{S}$ to $\mathbb{C}$!!) is denoted as $\langle F, \phi \rangle = F(\phi) = \int F(x)\phi(x) \, \mathrm{d}x$.

- An example: *the Dirac delta function* $\delta(x) \in \mathcal{S}'$ is defined as $\langle \delta, \phi \rangle = \phi(0)$. In other words, $\displaystyle\int_{-\infty}^{\infty} \delta(x)\phi(x) \, \mathrm{d}x = \phi(0)$.

- For any $F \in \mathcal{S}'$ and any $\phi \in \mathcal{S}$, we can define the following operations:

  Differentiation: $\langle \partial_x^k F, \phi \rangle = (-1)^k \langle F, \partial_x^k \phi \rangle$.
  This can be shown by integration by parts. An example: $\langle \delta', \phi \rangle = -\langle \delta, \phi' \rangle = -\phi'(0)$.

  Convolution: $F * \phi(x) = \langle F, \tau_x \widetilde{\phi} \rangle$, where $\widetilde{\phi}(y) = \phi(-y)$.
  An example: $(\delta * \phi)(x) = \phi(x)$.

  Fourier transform: $\langle \hat{F}, \phi \rangle = \langle F, \hat{\phi} \rangle$.
  An example: $F = \delta$, then $\langle \hat{\delta}, \phi \rangle = \langle \delta, \hat{\phi} \rangle = \hat{\phi}(0)$. This essentially shows that $\hat{\delta}(\xi) \equiv 1$. Using the translation operator, we can also have $\mathcal{F}\{\delta(x - a)\} = \mathrm{e}^{-2\pi \mathrm{i} \xi a}$, and $\mathcal{F}\{\mathrm{e}^{-2\pi \mathrm{i} x a}\} = \delta(\xi + a)$.

- **Definition:** A tempered distribution F on $\mathbb{R}$ is called *periodic* with period A if $\langle F, \tau_A \phi \rangle = \langle F, \phi \rangle$ for all $\phi \in \mathcal{S}$. A sequence of tempered distributions $\{F_n\}$ is said to *converge temperately* to a tempered distribution F if $\langle F_n, \phi \rangle \to \langle F, \phi \rangle$ as $n \to \infty$ for all $\phi \in \mathcal{S}$. (See that all these operations and definitions are now moved to the *nice spouses* of F!)

- **[Theorem]** If F is a periodic tempered distribution, then F can be expanded in a temperately convergent Fourier series, $F(x) = \frac{1}{\sqrt{A}} \sum_{-\infty}^{\infty} \alpha_k \mathrm{e}^{2\pi \mathrm{i} kx/A}$, i.e., $\langle F, \phi \rangle = \sum_{-\infty}^{\infty} \alpha_k \langle \frac{1}{\sqrt{A}} \mathrm{e}^{2\pi \mathrm{i} k \cdot /A}, \phi \rangle$ for all $\phi \in \mathcal{S}$. Moreover, the coefficients α_k satisfy $\alpha_k \leq C(1 + |k|)^N$ for some $C, N \geq 0$. Conversely, if $\{\alpha_k\}$ is any sequence satisfying this estimate, the series $\frac{1}{\sqrt{A}} \sum_{-\infty}^{\infty} \alpha_k \mathrm{e}^{2\pi \mathrm{i} kx/A}$ converges temperately to a periodic tempered distribution.

- Define the *Shah function* (or *comb function*), $\mathrm{III}_A(x) = \sum_{k=-\infty}^{\infty} \delta(x - kA)$. The facts about this function:

  1. Since this is a periodic tempered distribution, we can expand it into the temperately convergent Fourier series; $\mathrm{III}_A(x) \sim \frac{1}{A} \sum_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} kx/A}$. Note that $\alpha_k \equiv 1/\sqrt{A}$ for all $k \in \mathbb{Z}$.
  2. $\mathcal{F}\{\mathrm{III}_A\}(\xi) = \frac{1}{A} \mathrm{III}_{1/A}(\xi) = \frac{1}{A} \sum_{-\infty}^{\infty} \delta(\xi - \frac{k}{A})$.

- Using the Shah function and its Fourier transform, we can see that the Fourier transform of the Fourier series of a periodic function on $[-A/2, A/2]$ as follows:

$$f(x) \sim \frac{1}{\sqrt{A}} \sum_{-\infty}^{\infty} \alpha_k \mathrm{e}^{2\pi \mathrm{i} kx/A} \xrightarrow{\mathcal{F}} \frac{1}{\sqrt{A}} \sum_{-\infty}^{\infty} \alpha_k \delta(\xi - \frac{k}{A}) \quad \text{i.e., line spectrum (discrete)}$$

  As you can see, as A gets large, we are doing the finer sampling in the frequency domain, i.e.,

$$\begin{array}{ccc} f \in L^2[-A/2, A/2] & \xrightarrow{\mathcal{F}} & \hat{f} \in L^2(\mathbb{R}) \\ * \text{ convolution} & \xrightarrow{\mathcal{F}} & \cdot \text{ multiplication} \\ \mathrm{III}_A(x) & \xrightarrow{\mathcal{F}} & (1/A)\, \mathrm{III}_{1/A}(\xi) \\ \Updownarrow & \xrightarrow{\mathcal{F}} & \Updownarrow \\ \text{Periodization with period } A & \xrightarrow{\mathcal{F}} & \text{Discretization with rate } 1/A \text{ and scaling with factor } 1/A \end{array}$$

- **Periodization of a function with compact support $\Longleftrightarrow$ Discretization in frequency domain (with amplitude rescaling)**

For the details of the facts in these notes, see [1, Chap. 9], [2, Chap. 9], [3, Chap. 1].

References

[1] G. B. FOLLAND, *Fourier Analysis and Its Applications*, Wadsworth & Brooks/Cole, 1992.

[2] ——, *Real Analysis: Modern Techniques and Their Applications*, John Wiley & Sons, Inc., 2nd ed., 1999.

[3] E. M. STEIN AND G. WEISS, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, 1971.