Lecture 1 : Overture

Applied & Computational Harmonic Analysis (ACHA) is an extremely *interdisciplinary* field!
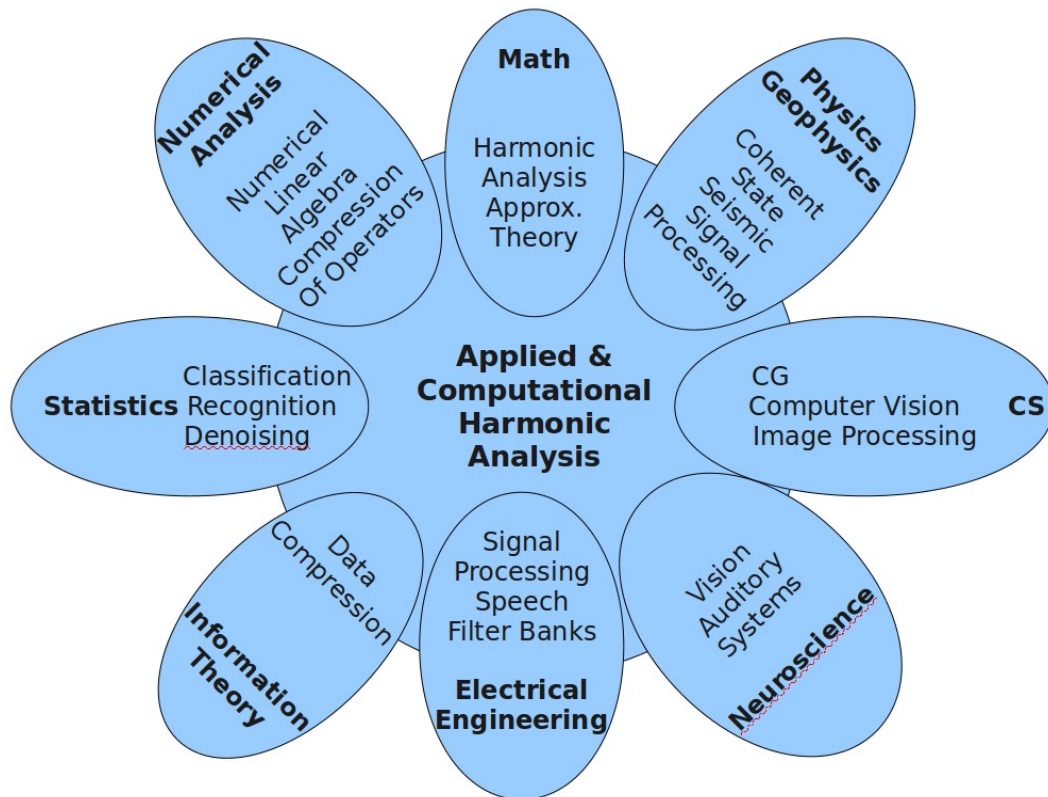
This is not surprising considering the ubiquity of the celebrated Fourier transform!

The key role of ACHA
→ provides great tools for representing your data/functions/operators.

* Representation of data/functions using ACHA-based tools is:

(1) Efficient → good for compression/approximation

$$f(x) \approx \sum_{k=1}^{n} \alpha_k \varphi_k(x)$$

where n: small, $\{\varphi_k\}$: an ONB
or more precisely,

$$\left\| f - \sum_{k=1}^{n} \alpha_k \varphi_k \right\| = O(n^{-\beta})$$
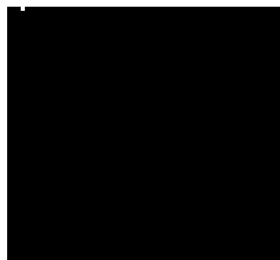
$$\exists \beta > 0$$

What classes of fcns possess such rate of approximation for a given set of $\{\varphi_k\}$?
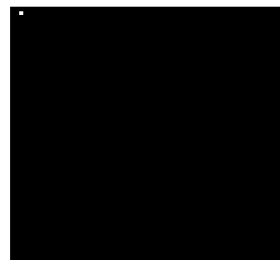
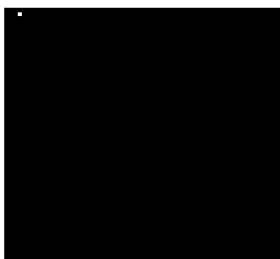A bad example:

Original   Basis #1   Basis #2

Basis #3   Approx with 1311 terms   Residual

A good example:

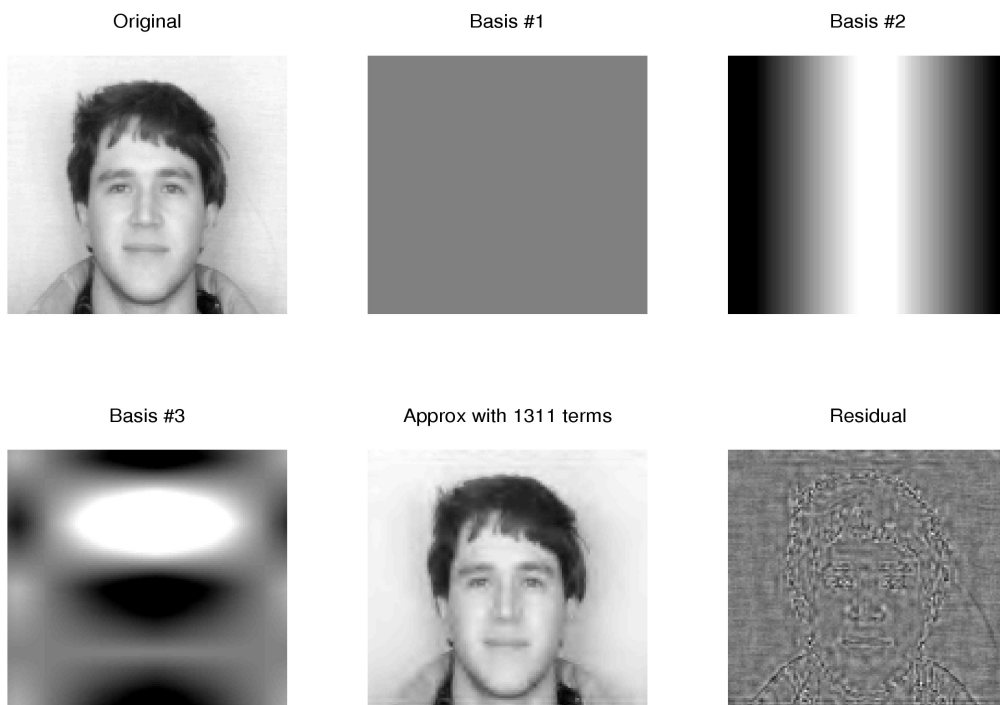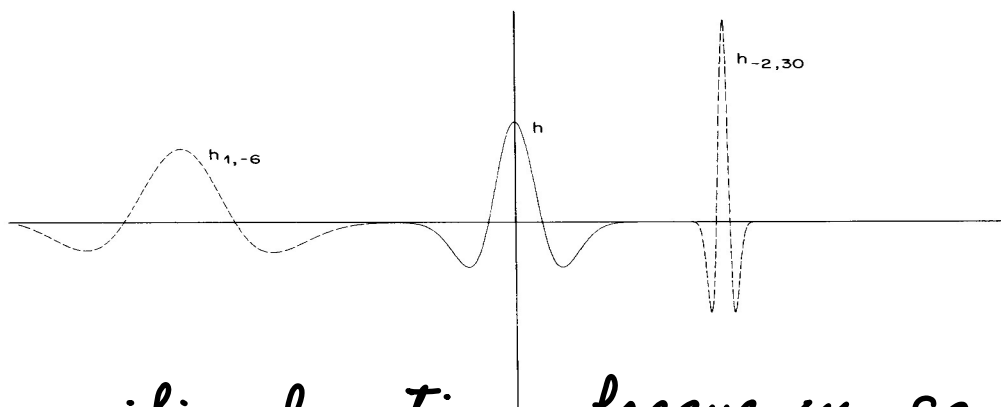| Original	Basis #1	Basis #2

Basis #3	Approx with 1311 terms	Residual

(2) **Meaningful/useful**

- $\{\alpha_k\}$ are used for interpretation, classification, discrimination, ...
- $\{\varphi_k\}$ are easily interpretable.

specific location, frequency, scale, duration, ...

Ex. 1.   Seismic waves

P   S   Surface wave
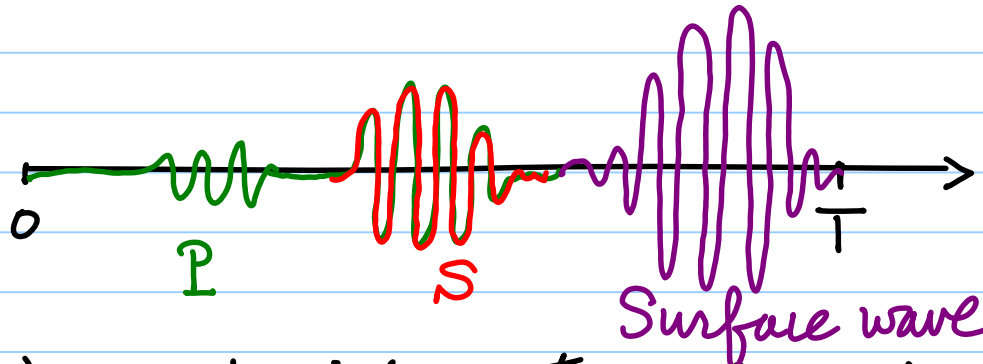
→ Wrong to interpret such a signal
with global sinusoids

Ex. 2   images (profiles of images)

cusps   sharp discontinuities
(discont. in derivatives)
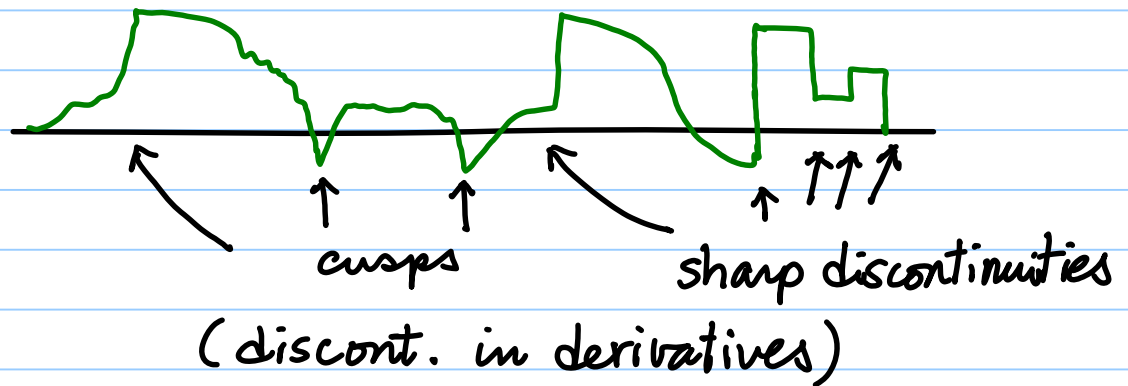
→ Wrong to use global sinusoids
                → less efficient,
                  coeff's do **not**
                  decay fast ...

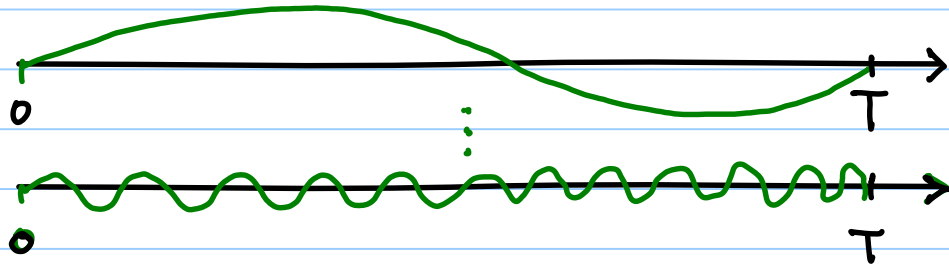(3) Representation should be *adapted* to your task!

Representation (or feature extraction) for compression may be different from that for classification
$\Rightarrow$ portraits of twins

(4) Computationally *fast* & *stable*

- If $\{\varphi_k\}$ are orthonormal, then
$$\alpha_k = \langle f, \varphi_k \rangle = \int f(x) \overline{\varphi_k(x)} \, dx$$
$\Rightarrow$ Matrix - vector multiplication in general.
$$\alpha = \underline{\Phi}^* f$$
which requires $O(n^2)$ op's : slow!
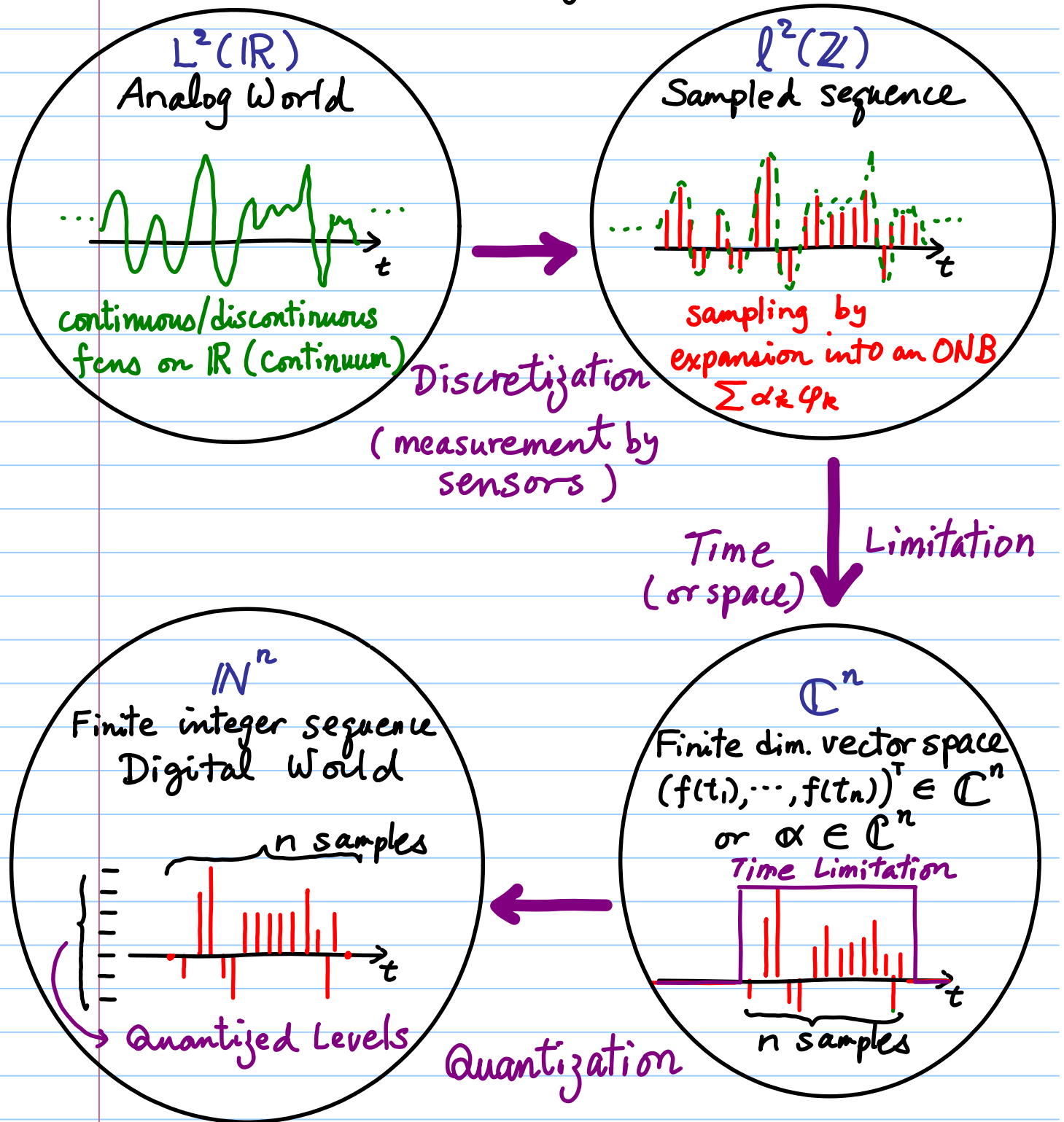Want to have $O(n \log n)$ or faster!

(5) Representation allows us to follow a very important strategy for any data / signal / image analysis tasks

It's a philosophy called

Ulf Grenander "*Analysis by Synthesis*"
Pattern Theory
We can truly "analyze" or "understand" the data / signal / images by looking at how they are composed of elementary "molecules" or "atoms".

* What is a signal?

$L^2(\mathbb{R})$
Analog World

continuous/discontinuous
fcns on $\mathbb{R}$ (continuum)

$\ell^2(\mathbb{Z})$
Sampled sequence

sampling by
expansion into an ONB
$\sum \alpha_k \varphi_k$

Discretization
(measurement by
sensors)

Time   Limitation
(or space)

$\mathbb{N}^n$
Finite integer sequence
Digital World

n samples

Quantized Levels

Quantization

$\mathbb{C}^n$
Finite dim. vector space
$(f(t_1), \cdots, f(t_n))^T \in \mathbb{C}^n$
or $\alpha \in \mathbb{C}^n$

Time Limitation

n samples

* We won't discuss the quantization procedure in this
* Sampling can be viewed as an expansion   course.
of a signal into a special ONB $\Rightarrow$ The Shannon Sampling Thm

* Until we discuss Harmonic Analysis on graphs & networks, we assume that our signals are supported on

$\mathbb{R}^d$, $d \geq 1$ (unbdd, continuum);

$\displaystyle\prod_{j=1}^{d} (a_j, b_j) \subset \mathbb{R}^d$ (a rectangle);

$\mathbb{Z}^d$, $d \geq 1$ (unbdd, discrete); or

$\displaystyle\prod_{j=1}^{d} \mathbb{Z}_{n_j}$, $n_j \in \mathbb{N}$ (regular lattice)

$\uparrow$

$(0, 1, \cdots, n_1 - 1) \times (0, 1, \cdots, n_2 - 1) \times \cdots \times (0, 1, \cdots, n_d - 1)$