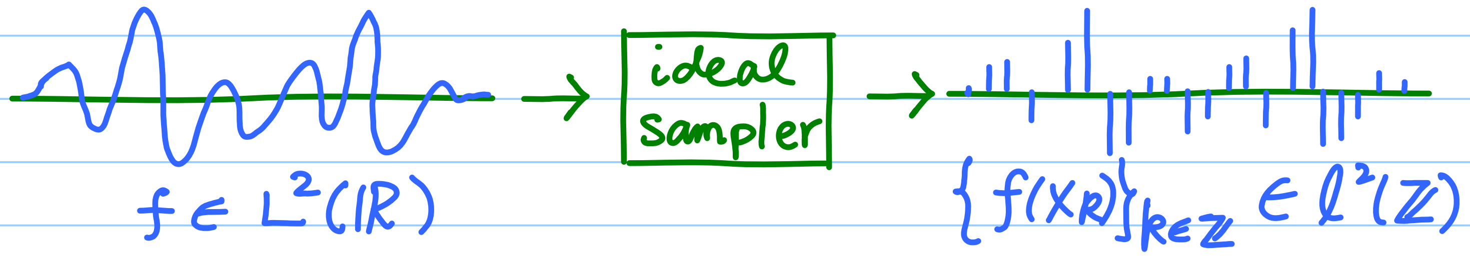
Lecture 4: Discretization via Sampling

Note Title

- The Riemann-Lebesgue Lemma says the high freq. components attenuates in L^1.
- We know from our daily experience that the high freq. info is difficult to transmit / propagate.

$\Rightarrow$ Band-limited fcns are common.



Bottom line question:

when we obtain sample values of $f(x)$ at $\cdots < x_{k-1} < x_k < x_{k+1} < \cdots$, how much information of f we retain or lose?

In general, $\{f(x_k)\}_{k\in\mathbb{Z}}$ do **not** tell anything about $f(x)$ for $x \notin \{x_k\}_{k\in\mathbb{Z}}$.

However, if $f \in$ a space of band-limited fcns $\subset L^2$, and $\{x_k\}_{k\in\mathbb{Z}}$ satisfy a certain condition, then this sequence $\{f(x_k)\}_{k\in\mathbb{Z}}$ tells you EVERYTHING about f!! Moreover, $\{f(x_k)\}_{k\in\mathbb{Z}}$ can be

expansion coeff's w.r.t. a specific ONB of the space of BL fcns.

Def. $BL_\Omega(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}) \;\middle|\; \hat{f}(\xi) = 0 \;\; \forall |\xi| \geq \frac{\Omega}{2} \right\}$.

Thm (The Sampling Thm)

Suppose $f \in BL_\Omega(\mathbb{R}) \subset L^2(\mathbb{R})$.
Then f is completely determined by its samples at $x = k/\Omega$, $k \in \mathbb{Z}$.
In fact,
$$f(x) = \sum_{k\in\mathbb{Z}} f(k/\Omega) \frac{\sin \pi\Omega(x - k/\Omega)}{\pi\Omega(x-k/\Omega)}.$$
$$= \sum_{k\in\mathbb{Z}} f(k/\Omega)\,\mathrm{sinc}(\Omega x - k)$$

Remark: This is a folklore thm. Often attributed to
E.T. Whittaker (1915), British → interpolation rather than sampling
V. Kotel'nikov (1933), Russian
H. Raabe (1939), German
C. Shannon (1948), American
I. Someya (1949), Japanese

But it turned out that K. Ogura (1920) seems to be the first who clearly stated the classical sampling thm. according to P.L. Butzer et al. (2011).

(Pf) Since $\hat{f}(\xi)$ is supported only on $[-\frac{\Omega}{2}, \frac{\Omega}{2}]$, we can expand it into the Fourier series:
$$\hat{f}(\xi) = \sum_{k\in\mathbb{Z}} \alpha_k \frac{1}{\sqrt{\Omega}} e^{2\pi i k\xi/\Omega}, \quad |\xi| \le \frac{\Omega}{2}.$$
$$\alpha_k = \langle \hat{f}, \frac{1}{\sqrt{\Omega}} e^{2\pi i k \xi/\Omega} \rangle$$
the k th Fourier coeff.
$$= \frac{1}{\sqrt{\Omega}} \int_{-\Omega/2}^{\Omega/2} \hat{f}(\xi)\, e^{-2\pi i k\xi/\Omega}\, d\xi$$

$$\overset{f \in BL_\Omega(\mathbb{R})}{=} \frac{1}{\sqrt{\Omega}} \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{-2\pi i k \xi/\Omega}\, d\xi$$

Fourier inversion thm!

$$= \frac{1}{\sqrt{\Omega}} f\left(-\frac{k}{\Omega}\right) \qquad — (*)$$

$$\Rightarrow f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi$$

$$= \int_{-\Omega/2}^{\Omega/2} \underbrace{\sum_{k\in\mathbb{Z}} \alpha_k \frac{1}{\sqrt{\Omega}} e^{2\pi i \xi k/\Omega}}_{\text{convergent}} e^{2\pi i \xi x}\, d\xi$$

via (*) + exchange of $\int$ & Σ

$$= \sum_{k\in\mathbb{Z}} \frac{1}{\Omega} f\left(-\frac{k}{\Omega}\right) \int_{-\Omega/2}^{\Omega/2} e^{2\pi i \xi (x + k/\Omega)}\, d\xi$$

$$= \sum_{k\in\mathbb{Z}} \frac{1}{\Omega} f\left(-\frac{k}{\Omega}\right) \left[\frac{e^{2\pi i \xi (x+k/\Omega)}}{2\pi i (x + k/\Omega)} \right]_{-\Omega/2}^{\Omega/2}$$

$$= \sum_{k\in\mathbb{Z}} f\left(-\frac{k}{\Omega}\right) \frac{\sin \pi (\Omega x + k)}{\pi (\Omega x + k)}$$

$$\overset{\text{exact!}}{=} \sum_{k\in\mathbb{Z}} \underbrace{f\left(\frac{k}{\Omega}\right)}_{\propto \text{ expansion coef}} \underbrace{\frac{\sin \pi \Omega (x - k/\Omega)}{\pi \Omega (x - k/\Omega)}}_{\propto \text{ ONB of } BL_\Omega(\mathbb{R})} \quad (**) \quad ///$$

So <u>if</u> we sample points on $x_k = k/\Omega$, $k \in \mathbb{Z}$, <u>then</u> we completely know $f(x)$ not only at $x \in \{x_k\}_{k\in\mathbb{Z}}$, but $\forall x \in \mathbb{R}$!

To obtain $f(x)$ for $x \notin \{x_k\}_{k\in\mathbb{Z}}$, we use (**) above, which is called the band-limited interpolation formula.

Exercise: (1) Compute $\mathcal{F}[\operatorname{sinc}(\Omega x - k)]$.
(2) Show that $\{\sqrt{\Omega}\operatorname{sinc}(\Omega x - k)\}_{k\in\mathbb{Z}}$ form an ONB of $BL_\Omega(\mathbb{R})$

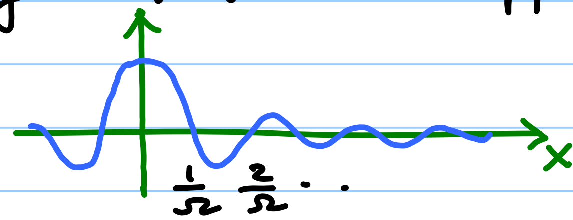
Remarks:

- Let $\Delta x_k := x_{k+1} - x_k = \frac{1}{\Omega} =: \Delta x$.

  the sampling rate or the sampling interval

  $\Omega \uparrow \iff \Delta x \downarrow$: quite intuitive.

- a problem of sinc fcn
  = decay is slow & support is not finite.



- $\frac{1}{\sqrt{\Omega}} f\left(\frac{k}{\Omega}\right) = \langle f, \sqrt{\Omega}\operatorname{sinc}(\Omega\cdot - k)\rangle$

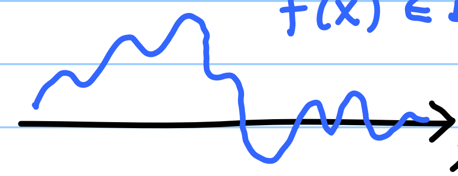
- $\sqrt{\Omega}\operatorname{sinc}(\Omega x - k) = \underbrace{\delta_{\Omega^{-1}}}_{\text{dilation}} \underbrace{\tau_k}_{\text{translation}} \operatorname{sinc}(x)$.

In order to interpret the essence of the Sampling Thm, we need the notion of the generalized fcns.

Graphical Interpretation of the Sampling Thm.

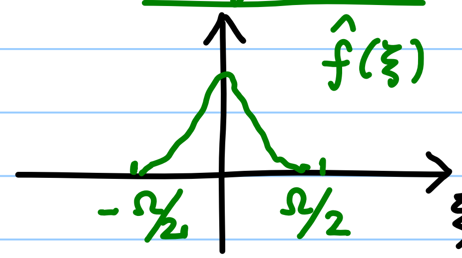
space / time | **frequency**

$f(x) \in BL_\Omega(\mathbb{R})$



$\xrightarrow{\mathcal{F}}$

$\hat{f}(\xi)$

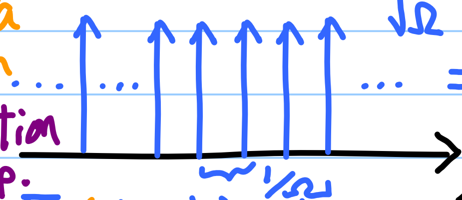


• multiplication | $*$ convolution

Dirac δ: delta fcn

δ_Ω^{-1}: dilation op.

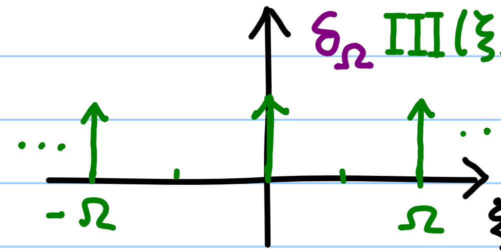
$\sqrt{\Omega}\sum \delta(\Omega x - k) = \delta_{\Omega^{-1}} \text{III}(x)$



$\text{III}(x) := \sum \delta(x-k)$, the shah (or comb)

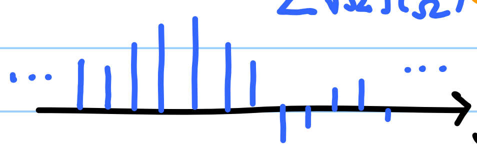
$\hat{\text{III}}(\xi) = \text{III}(\xi)$

$\delta_\Omega \text{III}(\xi)$



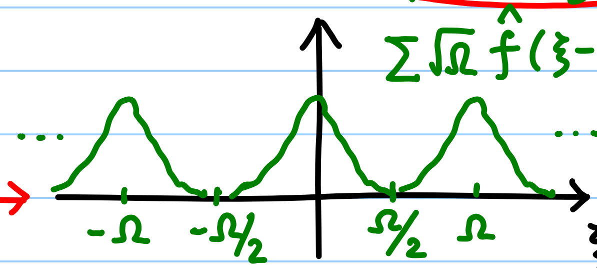
$\Downarrow$ sampling

$\sum \sqrt{\Omega} f(\tfrac{k}{\Omega}) \delta(\Omega x - k)$



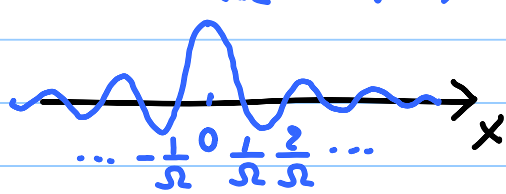
$\Downarrow$ periodization

$\sum \sqrt{\Omega}\, \hat{f}(\xi - k\Omega)$

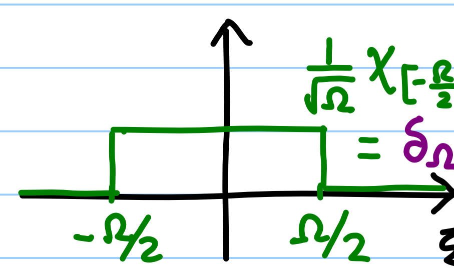


$*$ convolution | • multiplication

$\sqrt{\Omega}\,\text{sinc}(\Omega x) = \delta_{\Omega^{-1}} \text{sinc}(x)$

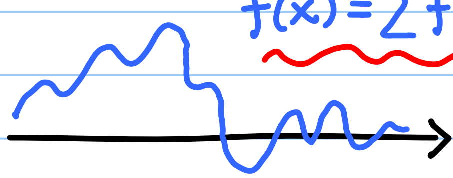


$\frac{1}{\sqrt{\Omega}} \chi_{[-\frac{\Omega}{2}, \frac{\Omega}{2}]}(\xi) = \delta_\Omega \chi_{[-\frac{1}{2}, \frac{1}{2}]}(\xi)$

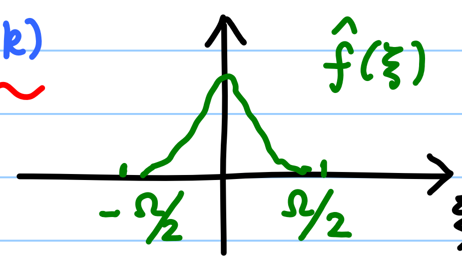


$\Downarrow$ | $\Downarrow$

$f(x) = \sum f(\tfrac{k}{\Omega})\,\text{sinc}(\Omega x - k)$



$\hat{f}(\xi)$



MAT 271: Applied & Computational Harmonic Analysis
Supplementary Notes I by Naoki Saito

The Generalized Functions

- The generalized functions have more singular behavior than functions (thus the name "generalized functions"), and are always defined as *linear functionals* on the *dual space*. Thus, before we discuss the generalized functions, we need to know the following.

  Definition: Let $\mathscr{X}$ be a vector space over, say, $\mathbb{C}$. A linear map from $\mathscr{X}$ to $\mathbb{C}$ is called a *linear functional* on $\mathscr{X}$. If $\mathscr{X}$ is a normed vector space, then the space $\mathscr{L}(\mathscr{X},\mathbb{C})$ of *bounded* linear functionals on $\mathscr{X}$ is called the *dual space*, and denoted by $\mathscr{X}^*$ (or $\mathscr{X}'$).
  Examples: The dual of $L^p(\mathbb{R})$, $1 < p < \infty$, is $L^q(\mathbb{R})$, where $(1/p) + (1/q) = 1$. These numbers are called *conjugate exponents*. In particular, L^2 is self dual. Similarly, the dual of the sequence space $\ell^p(\mathbb{Z})$ is $\ell^q(\mathbb{Z})$.

  Hölder's Inequality: Let p and q are conjugate exponents. Then for any $f \in L^p$, $g \in L^q$, we have

  $$\|fg\|_1 \le \|f\|_p \|g\|_q.$$

  (As you can see, the Cauchy-Schwarz inequality is a special version of this with $p = q = 2$. The proof is a great exercise.)

  The Riesz Representation Theorem: Suppose p and q are conjugate exponents with $1 < p < \infty$. Then for each linear functional $\varphi \in (L^p)^*$, there exists $g \in L^q$ such that $\varphi(f) = \int f(x)g(x)\,\mathrm{d}x$ for all $f \in L^p$. In other words, $(L^p)^*$ is isometrically isomorphic to L^q.

- The more singular the class of the generalized functions, the more regular its dual.

- We now define the *Schwartz class* $\mathscr{S} := \{f \in C^\infty(\mathbb{R}) \mid \sup_{x\in\mathbb{R}} |x^k \partial^\ell f| < \infty, \text{for any } k, \ell \in \mathbb{N}\}$, which are very smooth and decay faster than any polynomial at infinity, i.e., a very nice class of functions. An example: The Gaussian $g(x) = \mathrm{e}^{-x^2}$.

- Then, we consider the dual $\mathscr{S}'$. You can imagine that members of this class can be very singular or "spiky." This dual space is called the *tempered distributions*. Being as a linear functional, each member of $\mathscr{S}'$ acts on the Schwartz functions. More precisely, if $F \in \mathscr{S}'$ and $\phi \in \mathscr{S}$, then the value of F at ϕ (F is a *linear map* from $\mathscr{S}$ to $\mathbb{C}$!!) is denoted as $\langle F, \phi\rangle = F(\phi) = \int F(x)\phi(x)\,\mathrm{d}x$.

- An example: *the Dirac delta function* $\delta(x) \in \mathscr{S}'$ is defined as $\langle \delta, \phi\rangle = \phi(0)$. In other words, $\int_{-\infty}^{\infty} \delta(x)\phi(x)\,\mathrm{d}x = \phi(0)$.

- For any $F \in \mathscr{S}'$ and any $\phi \in \mathscr{S}$, we can define the following operations:

  Differentiation: $\langle \partial_x^k F, \phi\rangle = (-1)^k \langle F, \partial_x^k \phi\rangle$.
  This can be shown by integration by parts. An example: $\langle \delta', \phi\rangle = -\langle \delta, \phi'\rangle = -\phi'(0)$.

  Convolution: $F * \phi(x) = \langle F, \tau_x \widetilde{\phi}\rangle$, where $\widetilde{\phi}(y) = \phi(-y)$.
  An example: $(\delta * \phi)(x) = \phi(x)$.

  Fourier transform: $\langle \hat{F}, \phi\rangle = \langle F, \hat{\phi}\rangle$.
  An example: $F = \delta$, then $\langle \hat{\delta}, \phi\rangle = \langle \delta, \hat{\phi}\rangle = \hat{\phi}(0)$. This essentially shows that $\hat{\delta}(\xi) \equiv 1$. Using the translation operator, we can also have $\mathscr{F}\{\delta(x-a)\} = \mathrm{e}^{-2\pi\mathrm{i}\xi a}$, and $\mathscr{F}\{\mathrm{e}^{-2\pi\mathrm{i}xa}\} = \delta(\xi + a)$.

- **Definition:** A tempered distribution F on $\mathbb{R}$ is called *periodic* with period A if $\langle F, \tau_A \phi \rangle = \langle F, \phi \rangle$ for all $\phi \in \mathscr{S}$. A sequence of tempered distributions $\{F_n\}$ is said to *converge temperately* to a tempered distribution F if $\langle F_n, \phi \rangle \to \langle F, \phi \rangle$ as $n \to \infty$ for all $\phi \in \mathscr{S}$. (See that all these operations and definitions are now moved to the *nice spouses* of F!)

- **[Theorem]** If F is a periodic tempered distribution, then F can be expanded in a temperately convergent Fourier series, $F(x) = \frac{1}{\sqrt{A}} \sum_{-\infty}^{\infty} \alpha_k e^{2\pi i k x / A}$, i.e., $\langle F, \phi \rangle = \sum_{-\infty}^{\infty} \alpha_k \left\langle \frac{1}{\sqrt{A}} e^{2\pi i k \cdot / A}, \phi \right\rangle$ for all $\phi \in \mathscr{S}$. Moreover, the coefficients α_k satisfy $\alpha_k \le C(1+|k|)^N$ for some $C, N \ge 0$. Conversely, if $\{\alpha_k\}$ is any sequence satisfying this estimate, the series $\frac{1}{\sqrt{A}} \sum_{-\infty}^{\infty} \alpha_k e^{2\pi i k x / A}$ converges temperately to a periodic tempered distribution.

- Define the *Shah function* (or *comb function*), $\mathrm{III}_A(x) := \sum_{k=-\infty}^{\infty} \delta(x - kA)$. The facts about this function:

  1. Since this is a periodic tempered distribution, we can expand it into the temperately convergent Fourier series; $\mathrm{III}_A(x) \sim \frac{1}{A} \sum_{-\infty}^{\infty} e^{2\pi i k x / A}$. Note that $\alpha_k \equiv 1/\sqrt{A}$ for all $k \in \mathbb{Z}$.
  2. $\mathscr{F}\{\mathrm{III}_A\}(\xi) = \frac{1}{A} \mathrm{III}_{1/A}(\xi) = \frac{1}{A} \sum_{-\infty}^{\infty} \delta(\xi - \frac{k}{A})$.

- Using the Shah function and its Fourier transform, we can see that the Fourier transform of the Fourier series of a periodic function on $[-A/2, A/2]$ as follows:

$$f(x) \sim \frac{1}{\sqrt{A}} \sum_{-\infty}^{\infty} \alpha_k e^{2\pi i k x / A} \xrightarrow{\mathscr{F}} \frac{1}{\sqrt{A}} \sum_{-\infty}^{\infty} \alpha_k \delta(\xi - \frac{k}{A}) \quad \text{i.e., line spectrum (discrete)}$$

  As you can see, as A gets large, we are doing the finer sampling in the frequency domain, i.e.,

$$\begin{array}{ccc}
f \in L^2[-A/2, A/2] & \xrightarrow{\mathscr{F}} & \hat{f} \in L^2(\mathbb{R}) \\
* \text{ convolution} & \xrightarrow{\mathscr{F}} & \cdot \text{ multiplication} \\
\mathrm{III}_A(x) & \xrightarrow{\mathscr{F}} & (1/A)\,\mathrm{III}_{1/A}(\xi) \\
\Updownarrow & \xrightarrow{\mathscr{F}} & \Updownarrow \\
\text{Periodization with period } A & \xrightarrow{\mathscr{F}} & \text{Discretization with rate } 1/A \text{ and scaling with factor } 1/A
\end{array}$$

- **Periodization of a function with compact support $\Longleftrightarrow$ Discretization in frequency domain (with amplitude rescaling)**

For the details of the facts in these notes, see [1, Chap. 9], [2, Chap. 9], [3, Chap. 1].

References

[1] G. B. FOLLAND, *Fourier Analysis and Its Applications*, Amer. Math. Soc., Providence, RI, 1992. Republished by AMS, 2009.

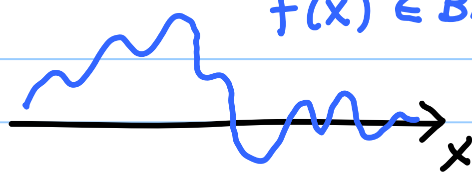
[2] ——, *Real Analysis: Modern Techniques and Their Applications*, John Wiley & Sons, Inc., 2nd ed., 1999.

[3] E. M. STEIN AND G. WEISS, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, 1971.

Reprise!

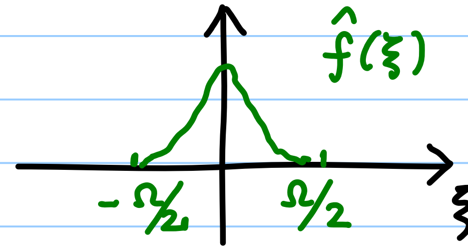
space/time | **frequency**

$f(x) \in BL_\Omega(\mathbb{R})$



$\xrightarrow{\mathcal{F}}$

$\hat{f}(\xi)$



$\cdot$ multiplication

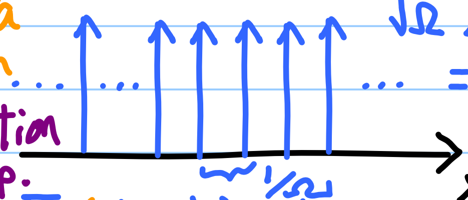
$*$ convolution

Dirac

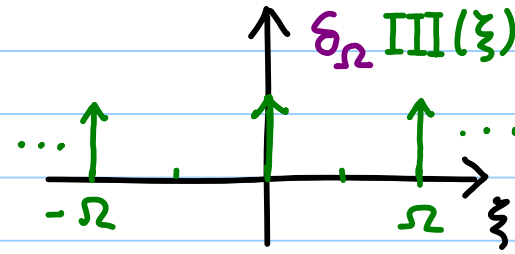
δ: delta fcn

$\delta_{\Omega^{-1}}$: dilation op.

$\sqrt{\Omega}\sum \delta(\Omega x - k) = \delta_{\Omega^{-1}}\,\mathrm{III}(x)$



$\delta_\Omega\,\mathrm{III}(\xi)$

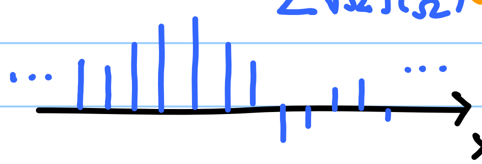


$\mathrm{III}(x) := \sum \delta(x-k)$, the shah (or comb)

$\hat{\mathrm{III}}(\xi) = \mathrm{III}(\xi)$

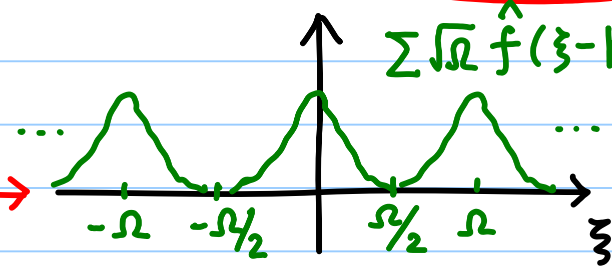
$\Downarrow$ sampling

$\sum \sqrt{\Omega} f(\frac{k}{\Omega})\,\delta(\Omega x - k)$



$\Downarrow$ periodization

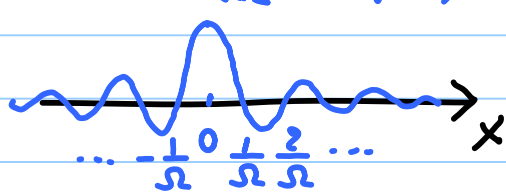
$\sum \sqrt{\Omega}\,\hat{f}(\xi - k\Omega)$



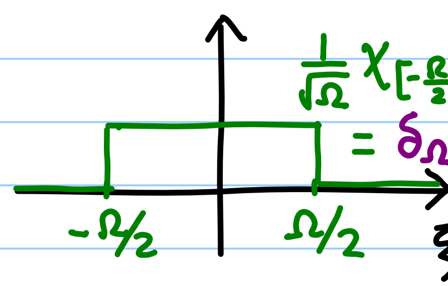
$*$ convolution

$\cdot$ multiplication

$\sqrt{\Omega}\,\mathrm{sinc}(\Omega x) = \delta_{\Omega^{-1}}\,\mathrm{sinc}(x)$



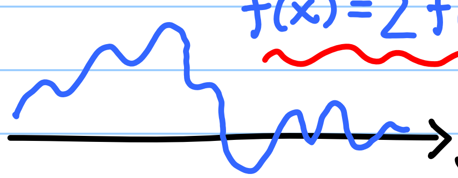
$\frac{1}{\sqrt{\Omega}}\chi_{[-\frac{\Omega}{2},\frac{\Omega}{2}]}(\xi) = \delta_\Omega\,\chi_{[-\frac{1}{2},\frac{1}{2}]}(\xi)$



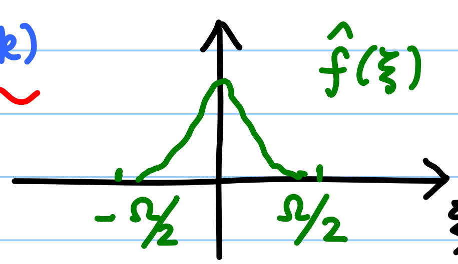
$\Downarrow$

$\Downarrow$

$f(x) = \sum f(\frac{k}{\Omega})\,\mathrm{sinc}(\Omega x - k)$



$\hat{f}(\xi)$



☆ The Sampling Thm via Poisson Summation Formula

See, e.g., the Wikipedia page & R.P. Boas's paper.

Poisson Summation Formula: $\forall f \in L^1(\mathbb{R})$,

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k)$$

So, we have

$$(*) \quad \sum \hat{f}(\xi+k) = \underbrace{\sum f(n)\, e^{-2\pi i n \xi}}_{\text{Fourier series with period 1}}$$

Suppose that $\hat{f} \in L^1(\mathbb{R})$.

Then $$\infty > \int_{-\infty}^{\infty} |\hat{f}(\xi)|\, d\xi = \sum_{k=-\infty}^{\infty} \int_k^{k+1} |\hat{f}(\xi)|\, d\xi$$

$$= \sum_{-\infty}^{\infty} \int_0^1 |\hat{f}(\xi+k)|\, d\xi$$

So, the above Fourier series converges to an $L^1(\mathbb{R})$ function whose nth Fourier coefficients

$$\int_0^1 \sum_{k=-\infty}^{\infty} \hat{f}(\xi+k)\, e^{2\pi i n \xi}\, d\xi = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i n \xi}\, d\xi = f(n)$$

So the series on the RHS of $(*)$ = the Fourier series of the LHS of $(*)$!

Multiply both sides of $(*)$ by $e^{2\pi i \xi x}$ and integrate in ξ from $-\frac{1}{2}$ to $\frac{1}{2}$:

$$\sum_k \int_{-1/2}^{1/2} e^{2\pi i \xi x} \hat{f}(\xi+k)\, d\xi = \sum_n f(n) \int_{-1/2}^{1/2} e^{2\pi i \xi (x-n)}\, d\xi$$

If $\operatorname{supp} \hat{f} = [-\frac{1}{2}, \frac{1}{2}]$, then $\quad = \sum_n f(n)\, \operatorname{sinc}(x-n)$

we can show: the LHS $= \int_{-1/2}^{1/2} \hat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi = f(x)$ ///