Lecture 5: Fourier Series on Intervals

Note Title

Some ways to convert an L^2 fcn into a sequence:

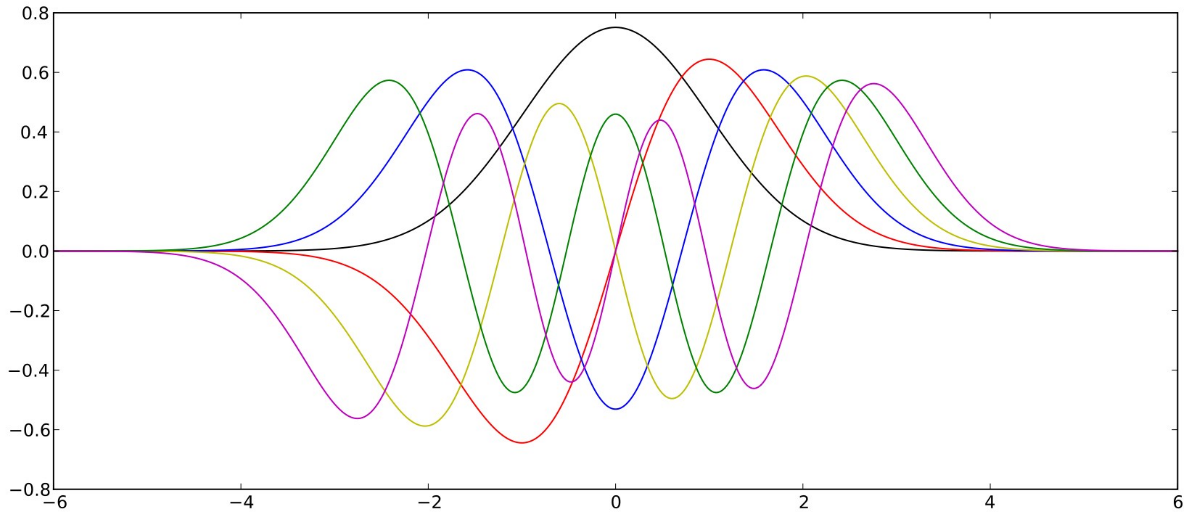
(1) If supp(f) is compact (i.e., time/space-limited)
$\Rightarrow$ Fourier series

(2) If supp($\hat{f}$) is compact (i.e., band-limited)
$\Rightarrow$ Sampling

(3) Otherwise, can use wavelets, local cosines etc.

(4) Another possibility: Hermite fcns (slow though)

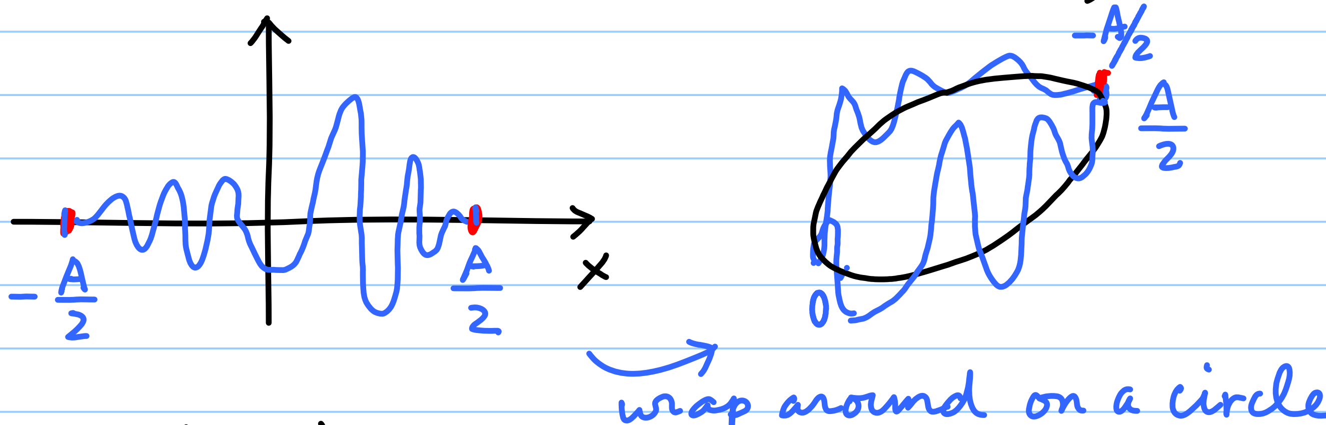
$$\psi_n(x) = (-1)^n (2^n n! \sqrt{\pi})^{-1/2} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2}$$



(5) Other methods ...

We already discussed (2).
Now let's discuss (1): Fourier Series!

W.L.O.G., let's consider $f \in L^2[-\frac{A}{2}, \frac{A}{2}]$ (since we can always shift a given fcn with a compact support to $[-\frac{A}{2}, \frac{A}{2}]$.)



Consider this as a periodic fcn with period A.

<u>Thm</u> $\left\{ \frac{1}{\sqrt{A}} e^{2\pi i k x/A} \right\}_{k\in\mathbb{Z}}$ form an ONB of $L^2[-\frac{A}{2}, \frac{A}{2}]$.

<u>Note</u>: $\frac{1}{\sqrt{A}} e^{2\pi i k x/A} = \delta_A (e^{2\pi i k x})$

So $L^2[-\frac{A}{2}, \frac{A}{2}] \cong \delta_A (L^2[-\frac{1}{2}, \frac{1}{2}])$

Also note: This ONB can be written as

$$\left\{ \frac{1}{\sqrt{A}} \right\} \cup \left\{ \sqrt{\frac{2}{A}} \cos \frac{2\pi k}{A} x \right\}_{k\in\mathbb{N}} \cup \left\{ \sqrt{\frac{2}{A}} \sin \frac{2\pi k}{A} x \right\}_{k\in\mathbb{N}}$$

(Pf) Let $\varphi_k(x) := \frac{1}{\sqrt{A}} e^{2\pi i k x/A}$, $k \in \mathbb{Z}$.

Then it's easy to show $\{\varphi_k\}_{k\in\mathbb{Z}}$ is an ON set, i.e., $\langle \varphi_k, \varphi_\ell \rangle = \underbrace{\delta_{k\ell}}_{\text{Kronecker's delta}}$. ✓

The main issue is to prove its completeness.

Below is a sketch / idea of the proof.

Define

$$C_\#(-A/2, A/2) := \{ f \in C(-A/2, A/2) \mid f(-A/2) = f(A/2) \}$$

$$C_0(-A/2, A/2) := \{ f \in C(-A/2, A/2) \mid f(x) = 0 \text{ for } x \in {}^\exists \underbrace{N(-A/2) \cup N(A/2)}_{\text{neighborhood of } \pm A/2} \}$$

Then, clearly,

$$C_0 \subset C_\# \subset L^2$$

Things to show:

Step 1: C_0 is dense in L^2, hence $C_\#$ is also dense in L^2.

Step 2: Let $\mathcal{M} := \overline{\operatorname{span}\{\varphi_k\}}$

want to show $\mathcal{M} = L^2$.

But, it suffices to show $C_\# \subset \mathcal{M}$ thanks to Step 1.

To do so, one needs to show:

$\forall f \in C_\#$, $f \in \mathcal{M}$.

See, e.g., Dym & McKean (Chap. 1) for the details. ///

Now, we can safely write for $f \in L^2[-\frac{A}{2}, \frac{A}{2}]$,

$$f = \sum_{-\infty}^{\infty} \alpha_k \varphi_k \;, \quad \alpha_k = \langle f, \varphi_k \rangle$$

$$= \frac{1}{\sqrt{A}} \int_{-A/2}^{A/2} f(x)\, e^{-2\pi i k x/A} \, dx$$

$$= \hat{f}(k)$$

This notation, $\hat{f}(k)$, is justified considering the dual of the important "take-home" idea of Lecture 4:

"Periodization in $\left\{ \begin{matrix} \text{space} \\ \text{time} \end{matrix} \right\}$

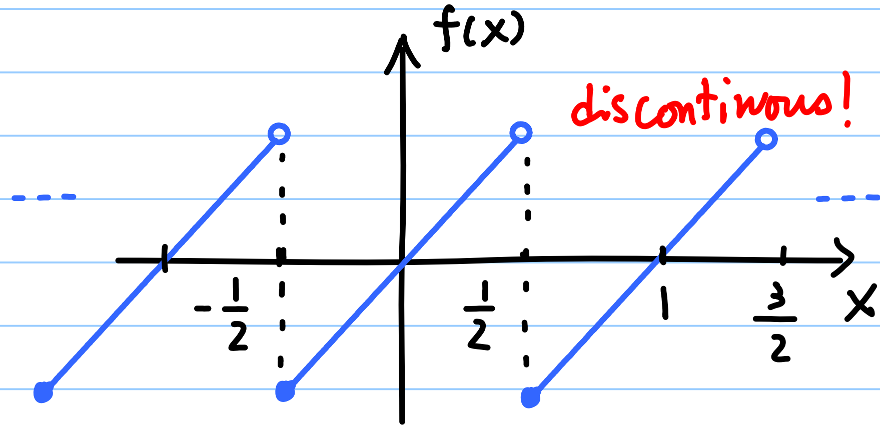
$\Longleftrightarrow$ Discretization in frequency."

See also my Note I, page 2.

✱ Examples of the Fourier Series:

(1) $f(x) = x, \quad -\frac{1}{2} \leq x < \frac{1}{2}$, i.e., $A = 1$.

$$\varphi_k(x) = e^{2\pi i k x}.$$



For $k \neq 0$, we have

$$\alpha_k = \int_{-\frac{1}{2}}^{\frac{1}{2}} x e^{-2\pi i k x}\, dx = \frac{x e^{-2\pi i k x}}{-2\pi i k}\Bigg|_{-\frac{1}{2}}^{\frac{1}{2}} + \frac{1}{2\pi i k}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i k x}\, dx$$

$$= \frac{\cos \pi k}{-2\pi i k} + \frac{1}{(4\pi k)^2}\, \cancel{e^{-2\pi i k x}}\Bigg|_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \frac{(-1)^k i}{2\pi k}$$

$$\alpha_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} x\, dx = 0 .$$

$k=0$ is excluded.

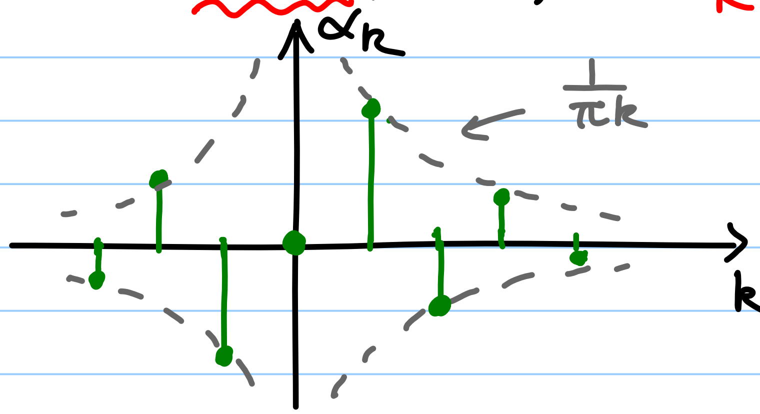
Hence $\underset{\text{1-periodic}}{f(x) = x} \sim \sum_k{}' \frac{(-1)^k i}{2\pi k} e^{2\pi i k x}$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\pi k} \sin(2\pi k x)$$

<u>Remarks</u>:

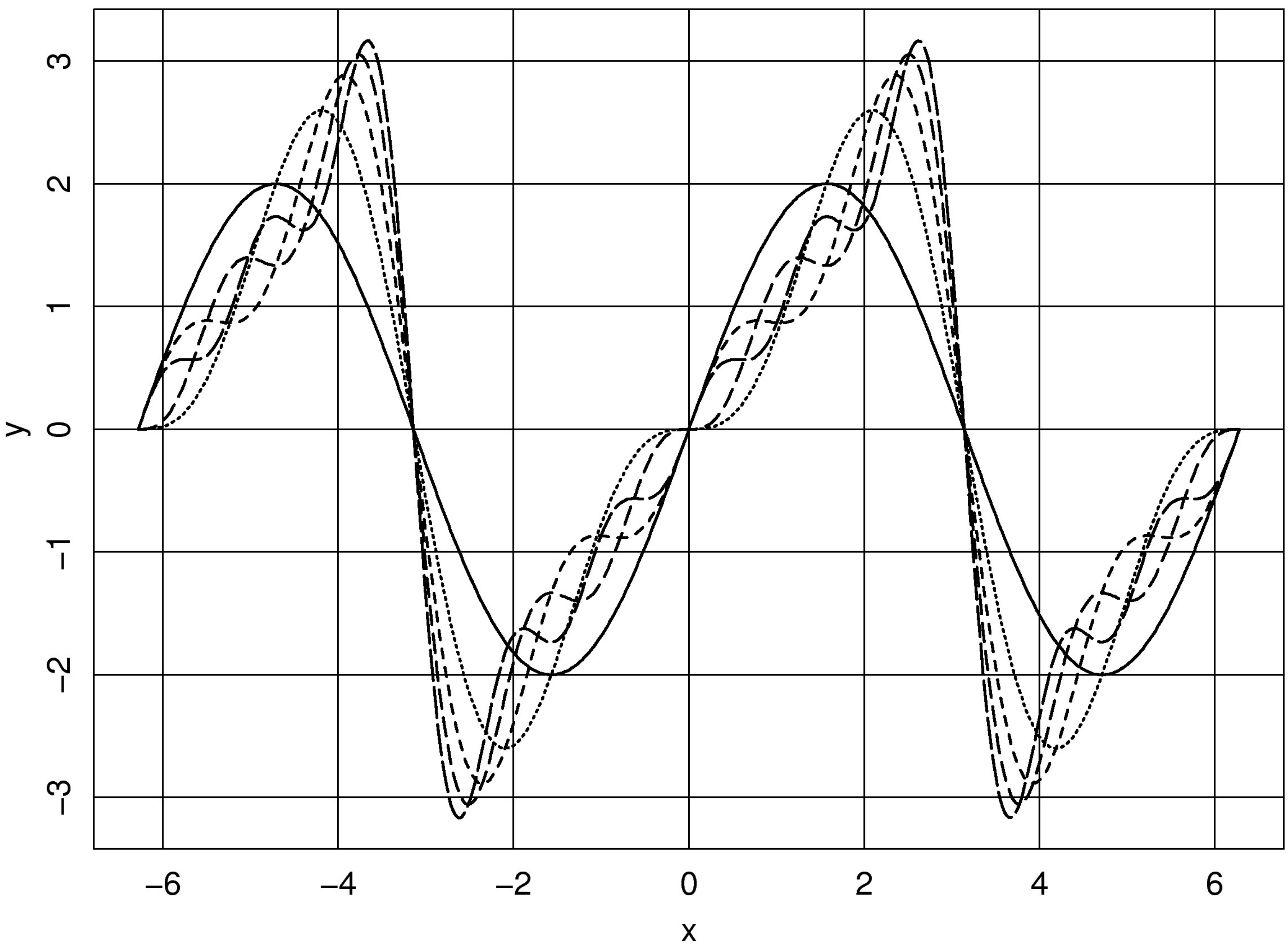
(1) $f(x) = x$ is an odd fcn.
Its Fourier series has only sine terms.
$\rightarrow$ makes sense.

(2) The decay of the Fourier coefficients is rather slow, i.e., $O(\frac{1}{k})$.

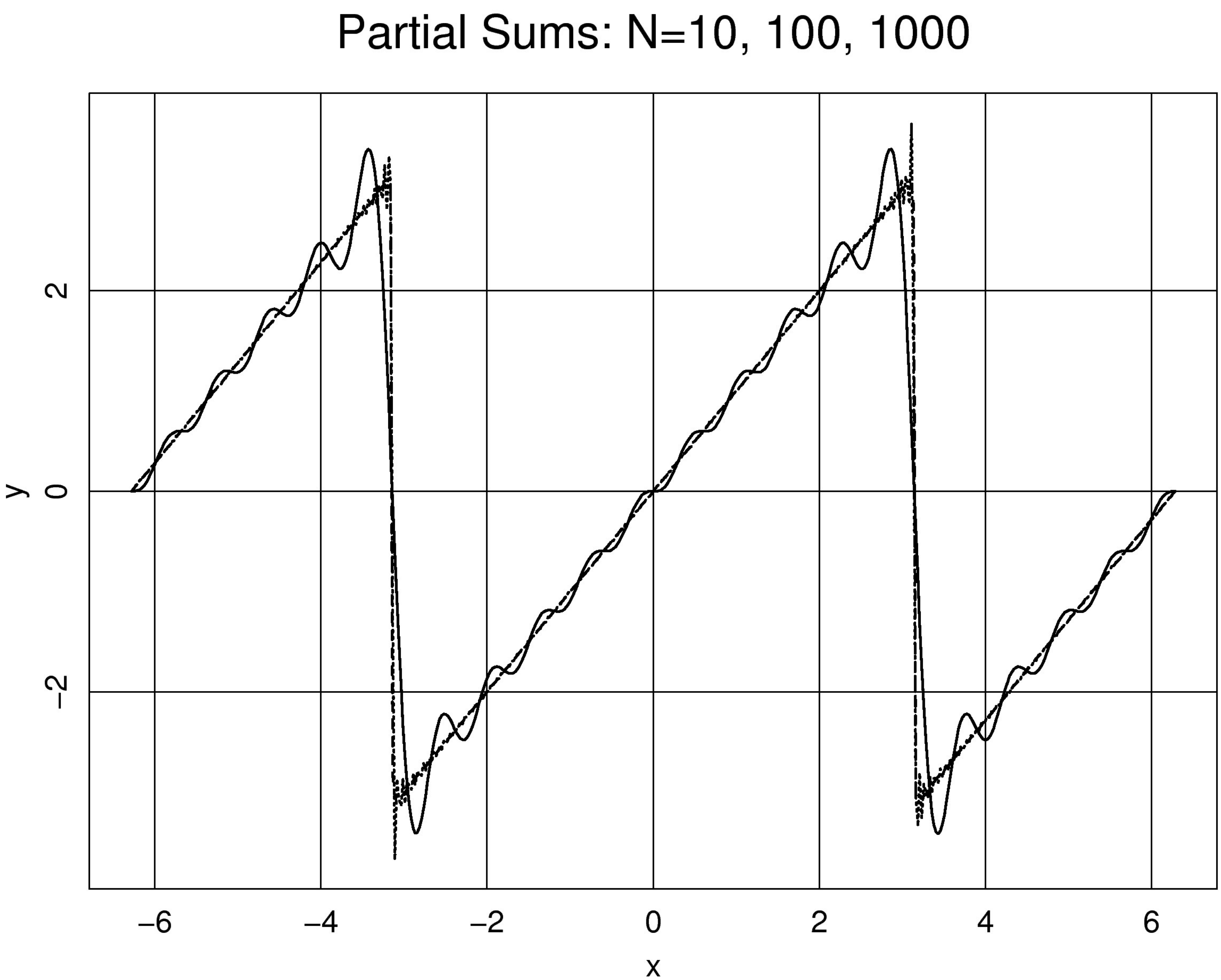


Let's look at the first few partial sums. (The interval was $[-\pi, \pi)$ instead of $[-\frac{1}{2}, \frac{1}{2})$.)

First 5 Partial Sums of the Fourier Series of f(x)=x (2*pi period



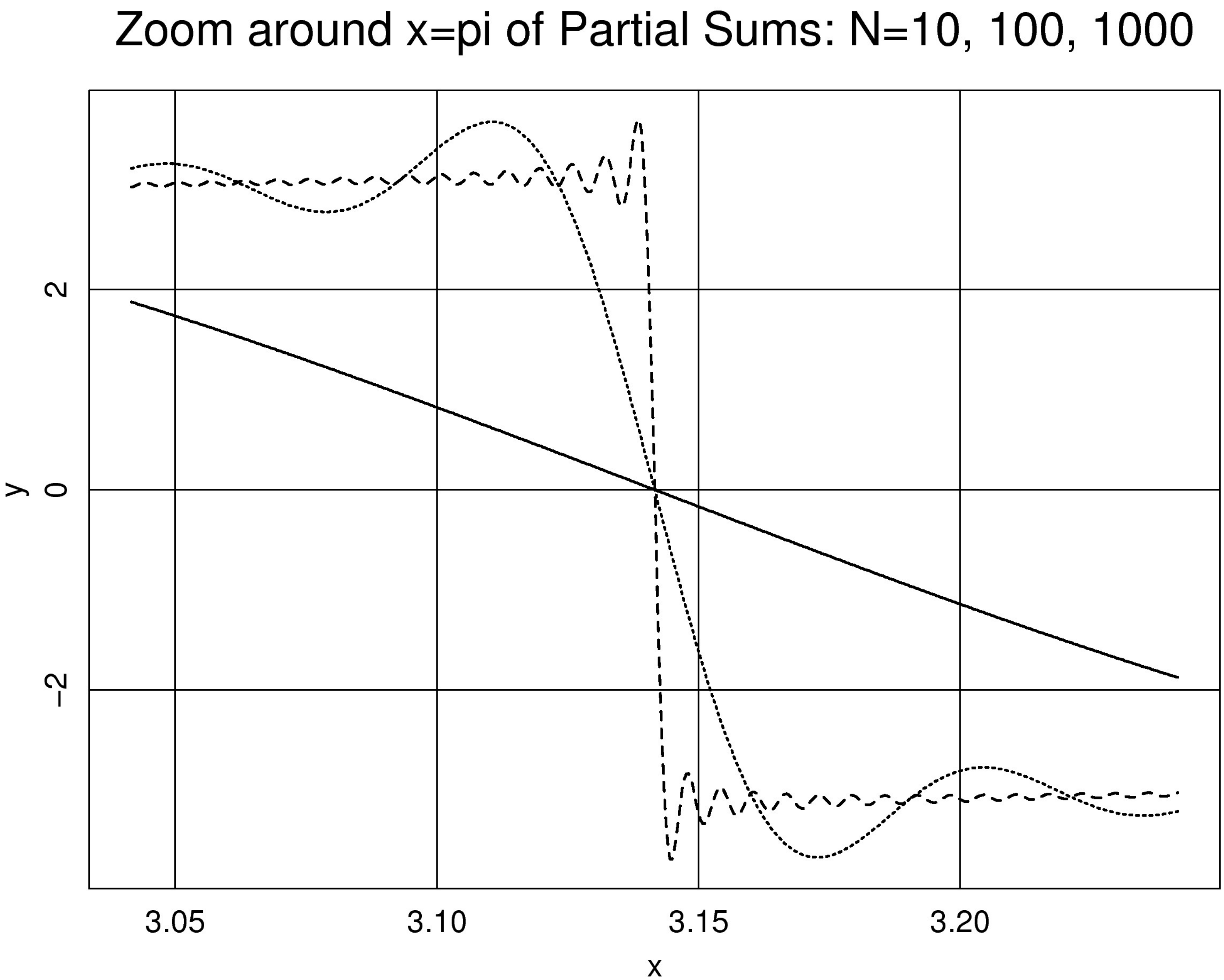
How about more terms ?



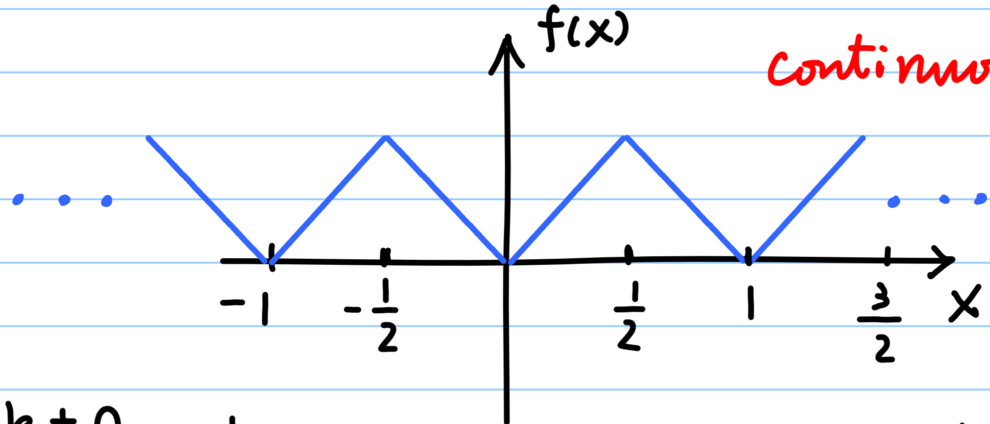
We still see the spurious oscillations around discontinuities.

⇒ This leads to the so-called Gibbs phenomenon.
(See Hewitt - Hewitt, 1979)

Zooming up around the discontinuity:



(2) $f(x) = |x| \ , \quad -\tfrac{1}{2} \le x < \tfrac{1}{2}.$



continuous! in C but not in C^1.

For $k \neq 0$,

$$\alpha_k = \int_{-\frac{1}{2}}^{\frac{1}{2}} |x| e^{-2\pi i k x} dx = 2\int_0^{\frac{1}{2}} x \cos 2\pi k x \, dx$$

$$= 2\left\{ \cancel{x \cdot \frac{\sin 2\pi k x}{2\pi k} \Big|_0^{\frac{1}{2}}} - \frac{1}{2\pi k}\int_0^{\frac{1}{2}} \sin 2\pi k x \, dx \right\}$$

$$= -\frac{1}{\pi k} \cdot \frac{-\cos 2\pi k x}{2\pi k}\Big|_0^{\frac{1}{2}} = \frac{\cos(\pi k) - 1}{2(\pi k)^2}$$

$$= \frac{(-1)^k - 1}{2\pi^2 k^2}$$

$$\alpha_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |x| \, dx = 2\int_0^{\frac{1}{2}} x \, dx = 2 \cdot \frac{x^2}{2}\Big|_0^{\frac{1}{2}} = \frac{1}{4}$$

So, $\underset{\text{1-periodic}}{f(x) = |x|} \sim \frac{1}{4} - \frac{1}{2}\sum_{k\in\mathbb{Z}}{}' \frac{1-(-1)^k}{\pi^2 k^2} e^{2\pi i k x}$

$$= \frac{1}{4} - \sum_{1}^{\infty} \frac{1-(-1)^k}{\pi^2 k^2} \cos(2\pi k x)$$

$$= \frac{1}{4} - 2\sum_{k=1}^{\infty} \frac{\cos(2\pi(2k-1)x)}{\pi^2 (2k-1)^2}$$

Remarks:

(1) The decay of the Fourier coeff. is faster: $O\left(\frac{1}{k^2}\right)$

(2) Evaluating this at $x=0$ leads to

$$f(0) = 0 = \frac{1}{4} - 2\sum_{1}^{\infty} \frac{1}{\pi^2(2k-1)^2}$$

$$\Leftrightarrow \quad \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$

So, we can also prove the celebrated

Basel problem: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$

(pf) $$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty}\left(\frac{1}{(2k-1)^2} + \frac{1}{(2k)^2}\right)$$

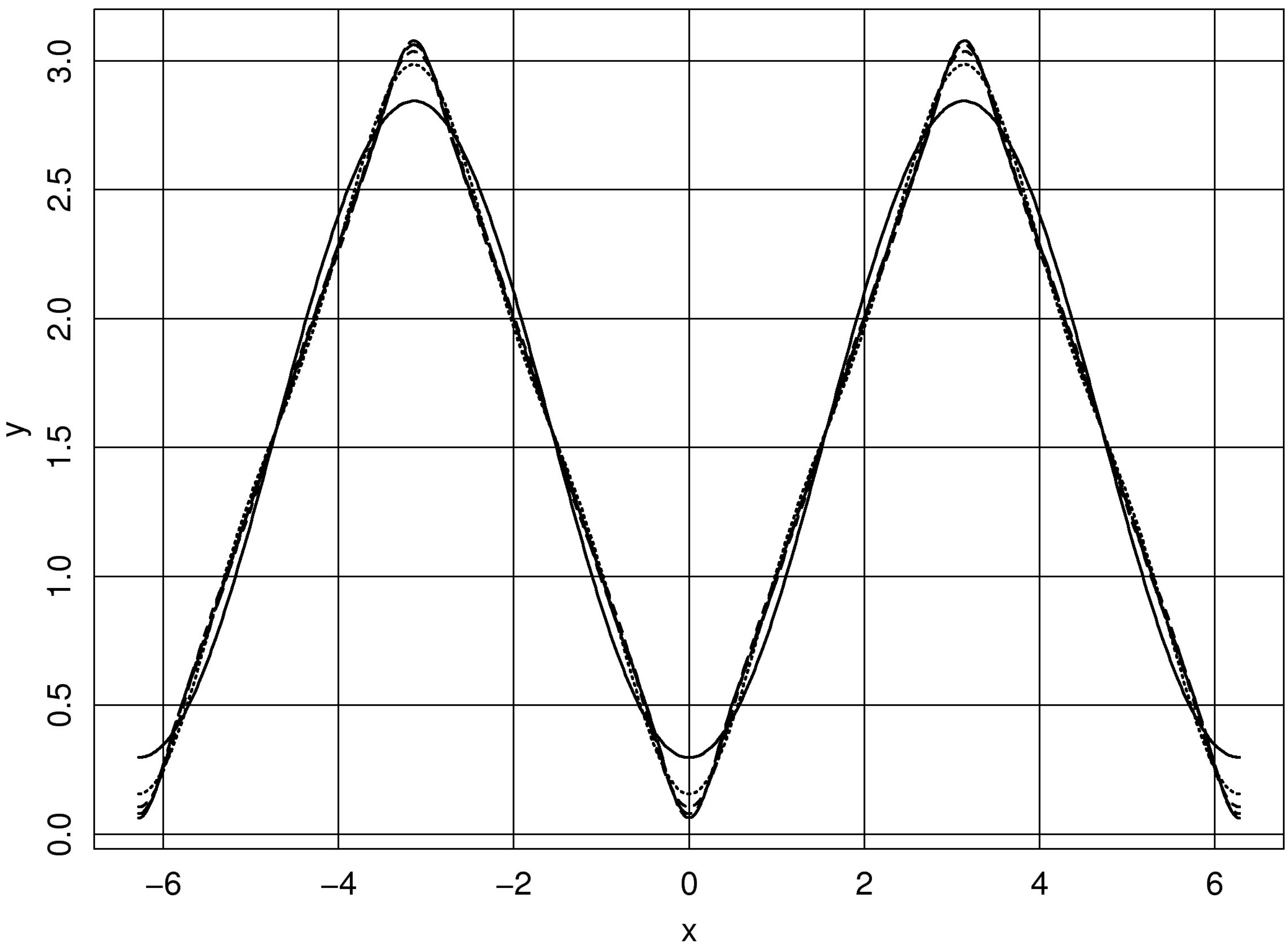
$$= \frac{\pi^2}{8} + \frac{1}{4}\sum_{1}^{\infty} \frac{1}{k^2}$$

$$\Leftrightarrow \quad \sum_{1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \quad ///$$

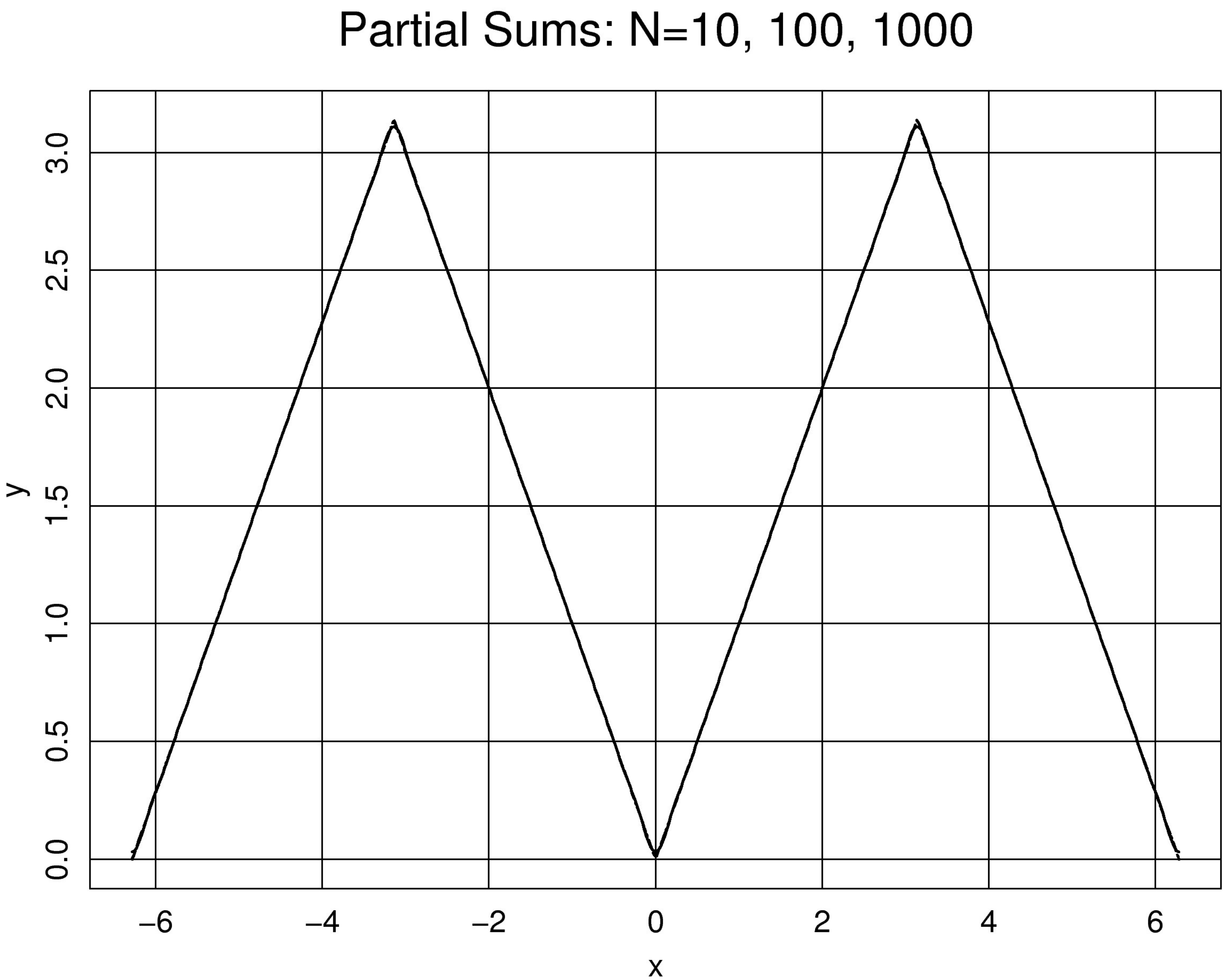
$\exists$ 8 or so ways to prove the Basel problem!!

How good is it to have faster decaying Fourier coefficients?

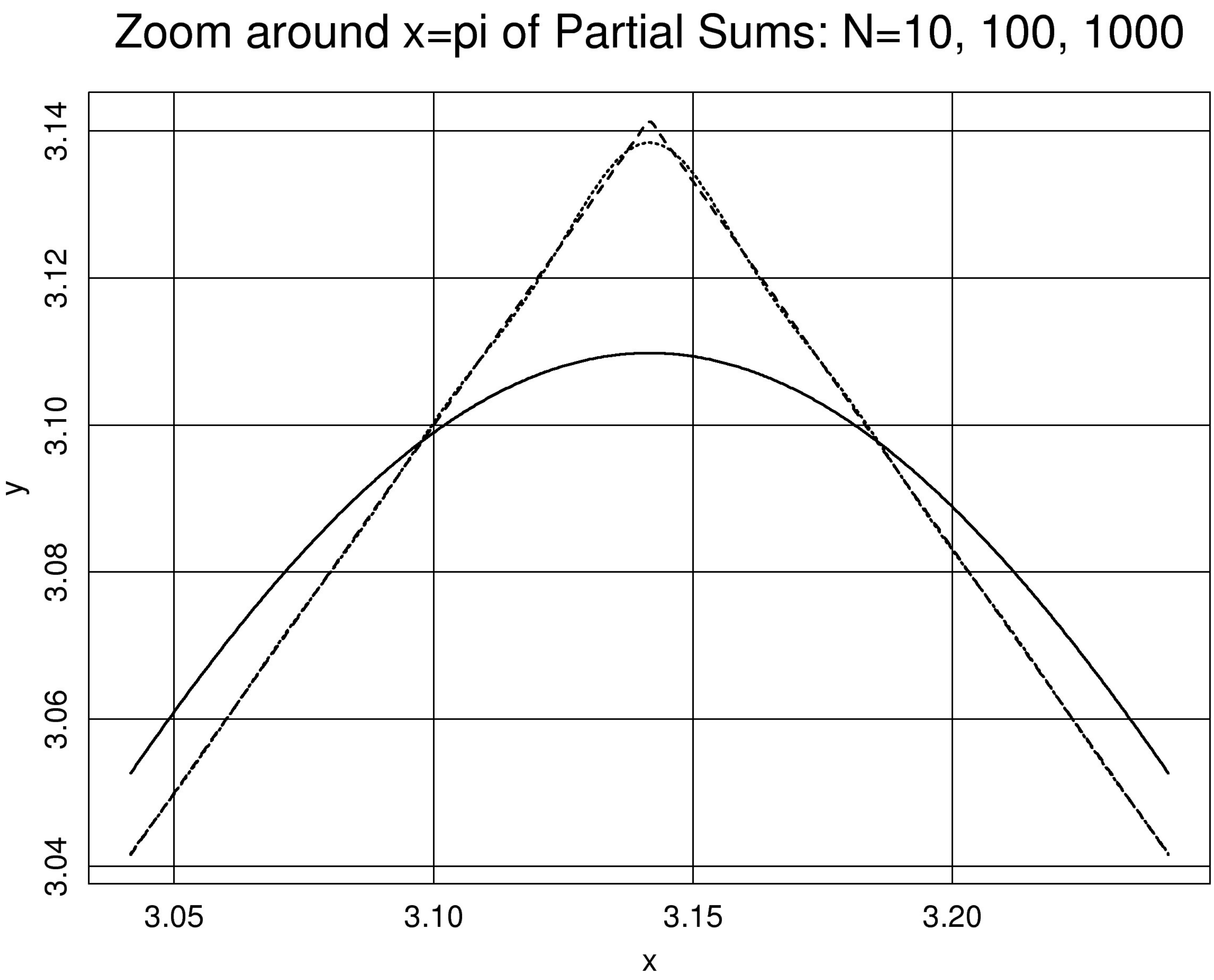
First 5 Partial Sums of the Fourier Series of f(x)=|x| (2*pi period



More terms?



Let's zoom up at the apex at $x = \pi$
where f' is discontinuous (but f is cont.)



MAT 271: Applied & Computational Harmonic Analysis: Supplementary Notes II by Naoki Saito

A Brief History of the Convergence of the Fourier Series

Theorem 1 (Dirichlet, 1829) Suppose f is 1-periodic, piecewise smooth on $\mathbb{R}$. Then, nth partial sum, $S_n[f](x) := \sum_{-n}^{n} c_k \mathrm{e}^{2\pi \mathrm{i} k x}$, satisfies

$$\lim_{n\to\infty} S_n[f](x) = \frac{1}{2}\left[f(x+) + f(x-)\right].$$

In particular, if x is a point of continuity, then $\lim_{n\to\infty} S_n[f](x) = f(x)$.

Theorem 2 (du Bois Reymond, 1876) There exists $f \in C(I)$ such that $\{S_n[f](0)\}$ diverges, where I is an interval of unit length.

Theorem 3 (A weak version of Fejér's Theorem) If f is 1-periodic, *continuous*, and piecewise smooth on $\mathbb{R}$, then the Fourier series of f converges to f *absolutely* and *uniformly*.

Definition: Suppose a series of functions $\sum_{1}^{\infty} g_n(x)$ converges to $g(x)$ on a set $x \in I$. Then, the convergence is called *absolute* if $\sum_{1}^{\infty} |g_n(x)|$ also converges for $x \in I$.

If we have $\sup_{x\in I} \left| g(x) - \sum_{1}^{N} g_n(x) \right| \to 0$ as $N \to \infty$, then we call this a *uniform* convergence.

Theorem 4 (Fejér 1904) If $f \in C(I)$, then the Cesàro means of $S_n[f]$ converge *uniformly* to f.

Definition: The mth *Cesàro mean* of partial sums is the mean of the first $m+1$ partial sums, i.e., $\sigma_m[f](x) := \dfrac{1}{m+1} \sum_{n=0}^{m} S_n[f](x)$.

Theorem 5 (Size of the Fourier coefficients and the smoothness of the functions) Suppose f is 1-periodic. If $f \in C^{k-1}(\mathbb{R})$ and $f^{(k-1)}$ is piecewise smooth (i.e., $f^{(k)}$ exists and piecewise continuous), then the Fourier coefficients of f, c_n, satisfy $\sum_n |n^k c_n|^2 < \infty$. In particular, $n^k c_n \to 0$. On the other hand, suppose $c_n, n \neq 0$, satisfy $|c_n| \le C|n|^{-(k+\gamma)}$ for some $C > 0$ and $\gamma > 1$. Then $f \in C^k(\mathbb{R})$.

Theorem 6 (Kolmogorov, 1926) There exists $f \in L^1(I)$ such that $\{S_n[f](x)\}$ diverges for every x.

Theorem 7 (Carleson, 1966) If $f \in L^2(I)$, then $S_n[f](x)$ converges to $f(x)$ almost everywhere.

Theorem 8 (Hunt, 1967) If $f \in L^p(I), p > 1$, then $S_n[f](x)$ converges to $f(x)$ almost everywhere.

Mathematicians are still trying to simplify the proof of the Carlson-Hunt theorem as of today.

For the details of the above facts, see [1, Chap. 1,2], [2, Chap. 1], [3, Part 1], and [4, Chap. 1]. [5, Chap. 1].

References

[1] J. M. ASH, ed., *Studies in Harmonic Analysis*, vol. 13 of MAA Studies in Mathematics, Math. Assoc. Amer., 1976.

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[3] T. W. KÖRNER, *Fourier Analysis*, Cambridge Univ. Press, 1988.

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[5] M. A. PINSKY, *Introduction to Fourier Analysis and Wavelets*, Amer. Math. Soc., Providence, RI, 2002. Republished by AMS, 2009.

☆ Smoothness Class Hierarchy

(from Davis & Rabinowitz: "Methods of Numerical Integration" Sec. 1.9, Dover 2007.)

From rough to smooth on an interval $[a,b]$ within the class of Riemann-integrable fcns.
↳ not including tempered distributions, e.g., δ, δ' etc.

- $R[a,b]$: bdd. & Riemann-integrable
- $BV[a,b]$: bdd. variation
- $PC[a,b]$: Piecewise-continuous
- $C[a,b]$: continuous
- $\mathrm{Lip}_\alpha[a,b]$: Lipschitz (or Hölder) continuous $\quad 0 < \alpha \le 1$.
- $C^1[a,b]$: continuously differentiable
- $C^n[a,b]$: n times cont. diff.
- $A(\Omega)$, $[a,b] \subset \Omega \subset \mathbb{C}$: analytic on Ω
- $E(\mathbb{C})$: entire (i.e., analytic on $\mathbb{C}$)

Def. Lipschitz (or Hölder) continuity.

$$\mathrm{Lip}_\alpha[a,b] := \{ f \in C[a,b] \mid |f(x) - f(y)| \le K \cdot |x-y|^\alpha, \ \exists K \ge 0, \ \forall x, y \in [a,b] \}.$$

Ex. $f(x) = \sqrt{x} \notin \mathrm{Lip}_\alpha[0,1]$ with $\frac{1}{2} < \alpha \le 1$, but
$\in \mathrm{Lip}_\alpha[0,1]$ with $0 < \alpha \le 1/2$

In the next lecture, we will discuss more about $BV[a,b]$, fcns of bdd. variation!