

Lecture 6: { Functions of Bounded Variation Fourier series on intervals II

Note Title

★ Functions of Bounded Variations

Why are we interested in fns of BVs?

- Often chosen as a model for piecewise smooth signals & images
- Useful in data compression & statistical estimation
- Provide sharp info on the decay rate of the Fourier coeff's.

Let $g(x)$ be a fn on a closed interval $I = [a, b]$. (I could be $\mathbb{R}$).

Let $D :=$ a subdivision of I , i.e.,
 $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$.

Now, let's form the sum:

$$T_D(g) := \sum_{k=1}^n |g(x_k) - g(x_{k-1})|$$

Def. If $T_D(g) < \infty$ for all possible subdivision D , then g is said to be of bdd. var. in I , and the total variation of g in I is defined as

$$V_I[g] = V_a^b[g] := \sup_D T_D(g).$$

$BV(I) :=$ a set of all fns of bdd. var. in I .

Fact: • $|g(b) - g(a)| \leq V_a^b[g] < \infty$ ← Take $x_0 = a$, $x_1 = b$.

• If $I \subset J$, then $V_I[g] \leq V_J[g]$

Thm 1. $g \in BV(I) \Rightarrow g$ is bdd. in I .

(Pf) $g(x) = g(a) + g(x) - g(a)$

$$\Rightarrow |g(x)| \leq |g(a)| + |g(x) - g(a)|$$

$$\leq |g(a)| + V_a^x[g]$$

$$\leq |g(a)| + V_a^b[g] < \infty. //$$

One can also show that $BV(I)$ is a
Banach space.

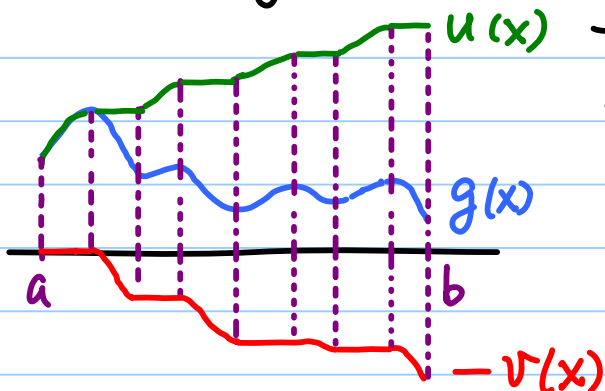
Thm 2. $g, h \in BV(I) \Rightarrow gh \in BV(I)$.
 $g, h \in BV(I), |h(x)| \geq \exists m > 0$
 $\Rightarrow g/h \in BV(I)$.

Thm 3. $\forall c \in (a, b), g \in BV[a, b]$
 $\Leftrightarrow g \in BV[a, c] \text{ \& } g \in BV[c, b]$.
Moreover, $V_a^b[g] = V_a^c[g] + V_c^b[g]$.

Remark: This can be generalized to
 $a < c_1 < c_2 < \dots < c_n < b$.

Thm 4 $g \in BV(I) \Leftrightarrow g$ can be written as
the difference of two
non decreasing fcn's.

say, $u(x) - v(x)$



M. Taibleson's Thm (1967) 1 page paper!

If $f \in BV[0,1]$, $f(x) \sim \sum_{-\infty}^{\infty} \alpha_k e^{2\pi i k x}$,

then $\alpha_k = O(1/k)$ as $k \rightarrow \infty$.

(Pf) Use the fact:

$$\int_{j/|k|}^{(j+1)/|k|} e^{-2\pi i k x} dx = 0, \quad k \neq 0, \quad j=0,1,\dots,|k|,$$

$$\odot \hookrightarrow = \frac{e^{-2\pi i (j+1) \frac{k}{|k|}} - e^{-2\pi i j \frac{k}{|k|}}}{-2\pi i k}$$

$$= \frac{1}{-2\pi i k} (e^{\mp 2\pi i (j+1)} - e^{\mp 2\pi i j}) = 0 //$$

Now, fix k , and let $a_j := \frac{j}{|k|}$, $j=0,1,\dots,|k|$.

Then define

$$g(x) := \sum_{j=0}^{|k|-1} f(a_j) \chi_{[a_j, a_{j+1}]}(x) \quad \text{a step fun approx. of } f.$$

Then, $\alpha_k[g] = \int_0^1 g(x) e^{-2\pi i k x} dx$

The k th Fourier coeff. of g .

$$= \int_0^1 \sum_{j=0}^{|k|-1} f(a_j) \chi_{[a_j, a_{j+1}]}(x) e^{-2\pi i k x} dx$$

$$= \sum_{j=0}^{|k|-1} f(a_j) \int_{a_j}^{a_{j+1}} e^{-2\pi i k x} dx = 0.$$

$$\alpha_k[f] = \int_0^1 f(x) e^{-2\pi i k x} dx \quad \text{with } \int_{a_j}^{a_{j+1}} e^{-2\pi i k x} dx = 0$$

$$|\alpha_k[f]| = |\alpha_k[f] - \alpha_k[g]| = |\alpha_k[f-g]|$$

$$= \left| \int_0^1 (f(x) - g(x)) e^{-2\pi i k x} dx \right|$$

$$\leq \sum_{j=0}^{|k|-1} \int_{a_j}^{a_{j+1}} |f(x) - f(a_j)| dx$$

$$\leq \sum_{j=0}^{|k|-1} V_{a_j}^{a_{j+1}}[f] \underbrace{(a_{j+1} - a_j)}_{= 1/|k|}$$

Thm 3 $\rightarrow$

$$= \frac{1}{|k|} \underbrace{V_0^1[f]}_{\text{"const."}} = O(1/|k|). \quad ///$$

Thm (NS - J.F. Remy, 2006)

Let f be a 1-periodic fcn and $f \in C^m(\mathbb{R})$. Furthermore, let us assume that $f^{(m+1)}$ exists and in $BV[0,1]$. Then its Fourier coeff.

$\alpha_k[f] = \hat{f}(k)$ decays as $O(|k|^{-m-2})$, where $m = 0, 1, \dots$.

(Pf) Use { the periodicity, i.e., $f^{(l)}(0) = f^{(l)}(1)$, $l = 0, \dots, m$.
integration by parts!

$$\begin{aligned} \hat{f}(k) &= \int_0^1 f(x) e^{-2\pi i k x} dx \\ &= \frac{e^{-2\pi i k x}}{-2\pi i k} f(x) \Big|_0^1 + \frac{1}{2\pi i k} \int_0^1 f'(x) e^{-2\pi i k x} dx \\ &= \frac{e^{-2\pi i k x}}{-(2\pi i k)^2} f'(x) \Big|_0^1 + \frac{1}{(2\pi i k)^2} \int_0^1 f''(x) e^{-2\pi i k x} dx \\ &= \dots = \frac{e^{-2\pi i k x}}{-(2\pi i k)^{m+1}} f^{(m)}(x) \Big|_0^1 + \frac{1}{(2\pi i k)^{m+1}} \int_0^1 f^{(m+1)}(x) e^{-2\pi i k x} dx \end{aligned}$$

By assumption, $f^{(m+1)} \in BV[0,1]$. So, can use the Taibleson Thm to get:

$$|\hat{f}(k)| \leq V_0^1[f^{(m+1)}] (2\pi)^{-m-1} |k|^{-m-2} \quad ///$$

★ Fourier Series on Intervals II

Suppose your fcn is defined on $[0, \frac{A}{2}]$ instead of $[-\frac{A}{2}, \frac{A}{2}]$.

$\Rightarrow \equiv$ two ways to make it an A -periodic fcn:

(1) Even extension

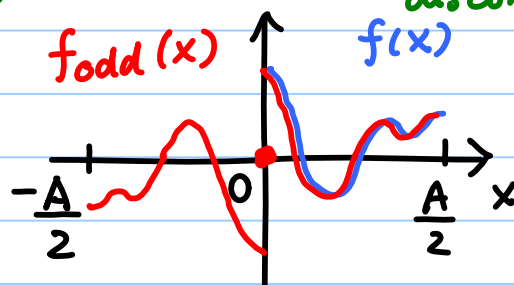
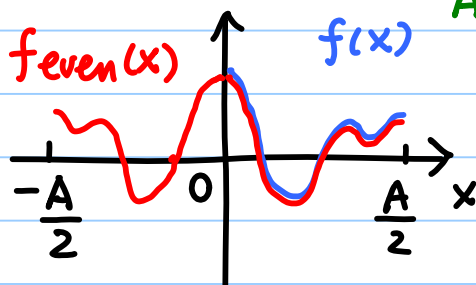
(2) Odd extension

$$f_{\text{even}}(x) = \begin{cases} f(x) & \text{if } x \in [0, \frac{A}{2}] \\ f(-x) & \text{if } x \in [-\frac{A}{2}, 0] \end{cases}$$

$$f_{\text{odd}}(x) = \begin{cases} f(x) & \text{if } x \in (0, \frac{A}{2}] \\ 0 & \text{if } x = 0 \\ -f(-x) & \text{if } x \in [-\frac{A}{2}, 0) \end{cases}$$

$f \in C[0, \frac{A}{2}] \Rightarrow f_{\text{even}} \in C(\mathbb{R})$
A-periodic

$f \in C[0, \frac{A}{2}] \Rightarrow f_{\text{odd}}:$
discontinuous



Then their Fourier series expansions get simpler.

But before computing them, let's review the relationship between the Fourier coefficients

$\{\alpha_k\}$ w.r.t. the ONB $\{\frac{1}{\sqrt{A}} e^{2\pi i k x/A}\}$ and $\{a_k, b_k\}$ w.r.t. the ONB $\{\frac{1}{\sqrt{A}}\} \cup \{\frac{\sqrt{2}}{\sqrt{A}} \cos(\frac{2\pi k x}{A})\} \cup \{\frac{\sqrt{2}}{\sqrt{A}} \sin(\frac{2\pi k x}{A})\}$.

Let $g(x)$ be an A -periodic L^2 fcn.

$$g(x) \sim \frac{1}{\sqrt{A}} \sum_{k \in \mathbb{Z}} \alpha_k e^{2\pi i k x/A}, \quad \alpha_k = \frac{1}{\sqrt{A}} \int_{-A/2}^{A/2} g(x) e^{-2\pi i k x/A} dx$$

$$\text{Then } \frac{1}{\sqrt{A}} \sum_{k \in \mathbb{Z}} \alpha_k e^{2\pi i k x/A}$$

$$= \frac{1}{\sqrt{A}} \left[\alpha_0 + \sum_{k=1}^{\infty} (\alpha_k + \alpha_{-k}) \cos\left(\frac{2\pi k x}{A}\right) + i(\alpha_k - \alpha_{-k}) \sin\left(\frac{2\pi k x}{A}\right) \right]$$

$$= \frac{\alpha_0}{\sqrt{A}} + \sum_1^{\infty} \left[\underbrace{\frac{\alpha_k + \alpha_{-k}}{\sqrt{2}}}_{a_k} \underbrace{\sqrt{\frac{2}{A}} \cos\left(\frac{2\pi k x}{A}\right)}_{\text{red wavy}} + \underbrace{\frac{\alpha_k - \alpha_{-k}}{\sqrt{2}} i}_{b_k} \underbrace{\sqrt{\frac{2}{A}} \sin\left(\frac{2\pi k x}{A}\right)}_{\text{red wavy}} \right]$$

If we want, we can write a_0 instead of α_0 .

Remark: In many books, the Fourier series is often defined on $[-\pi, \pi)$ and written as

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

where $a_0 = 2c_0$, $a_k = c_k + c_{-k}$, $b_k = i(c_k - c_{-k})$, $k \geq 1$.

But $\{e^{ikx}\}$ is an orthogonal basis of $L^2[-\pi, \pi]$, but not normalized. To make it an ONB, one needs the factor $\frac{1}{\sqrt{2\pi}}$.

The same can be said for the orthogonal basis $\{1\} \cup \{\cos kx\} \cup \{\sin kx\}$.

The ONB is $\{\frac{1}{\sqrt{2\pi}}\} \cup \{\frac{1}{\sqrt{\pi}} \cos kx\} \cup \{\frac{1}{\sqrt{\pi}} \sin kx\}$

i.e., $A = 2\pi$.

Compare this notation with mine in this lecture, which is the orthonormalized version:

$$\sum_{k=-\infty}^{\infty} \alpha_k \frac{1}{\sqrt{A}} e^{2\pi i k x / A} = \frac{1}{\sqrt{A}} a_0 + \sqrt{\frac{2}{A}} \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{2\pi k}{A} x\right) + b_k \sin\left(\frac{2\pi k}{A} x\right) \right]$$

where $a_0 = \alpha_0$, $a_k = \frac{\alpha_k + \alpha_{-k}}{\sqrt{2}}$, $b_k = \frac{\alpha_k - \alpha_{-k}}{\sqrt{2}} i$, $k \geq 1$.

Now, let's go back to the Fourier series expansion of f_{even} & f_{odd} .

$$\begin{aligned}\alpha_k[f_{\text{even}}] &= \frac{1}{\sqrt{A}} \int_{-A/2}^{A/2} f_{\text{even}}(x) e^{-2\pi i k x / A} dx \\ &= \frac{2}{\sqrt{A}} \int_0^{A/2} f(x) \cos\left(\frac{2\pi k x}{A}\right) dx \\ &= \alpha_{-k}[f_{\text{even}}] \quad \text{thanks to the evenness of } \cos \theta.\end{aligned}$$

Recall the relationship:

$$a_0 = \alpha_0, \quad a_k = \frac{\alpha_k + \alpha_{-k}}{\sqrt{2}}, \quad b_k = \frac{\alpha_k - \alpha_{-k}}{\sqrt{2}};$$

Hence, in this case of f_{even} , $a_0 = \alpha_0[f_{\text{even}}]$,
 $a_k = \sqrt{2} \alpha_k[f_{\text{even}}]$, $b_k \equiv 0$, $k \geq 1$.

In other words, f_{even} can be written as the Fourier **cosine** series:

$$f_{\text{even}}(x) \sim \frac{1}{\sqrt{A}} a_0 + \sum_{k=1}^{\infty} a_k \sqrt{\frac{2}{A}} \cos\left(\frac{2\pi k x}{A}\right)$$

$$\text{where } a_0 = \frac{2}{\sqrt{A}} \int_0^{A/2} f(x) dx = \alpha_0[f_{\text{even}}],$$

$$\begin{aligned}a_k &= 2 \sqrt{\frac{2}{A}} \int_0^{A/2} f(x) \cos\left(\frac{2\pi k x}{A}\right) dx \\ &= \sqrt{2} \alpha_k[f_{\text{even}}].\end{aligned}$$

Similarly, for $f_{\text{odd}}(x)$,

$$\alpha_k[f_{\text{odd}}] = \frac{1}{\sqrt{A}} \int_{-A/2}^{A/2} f_{\text{odd}}(x) e^{-2\pi i k x / A} dx$$

$$= \frac{-2i}{\sqrt{A}} \int_0^{A/2} f(x) \sin\left(\frac{2k\pi x}{A}\right) dx$$

$$= -\alpha_{-k}[f_{\text{odd}}] \text{ due to the oddness of } \sin \theta.$$

$$\Rightarrow a_k = \frac{\alpha_k + \alpha_{-k}}{\sqrt{2}} \equiv 0, \quad k \in \mathbb{N}.$$

$$a_0 = \alpha_0 = \frac{1}{\sqrt{A}} \int_{-A/2}^{A/2} f_{\text{odd}}(x) dx = 0.$$

No
cosines

$$b_k = \frac{\alpha_k - \alpha_{-k}}{\sqrt{2}} i = \sqrt{2} i \alpha_k[f_{\text{odd}}].$$

$$\text{So, } f_{\text{odd}}(x) \sim \sum_{k=1}^{\infty} b_k \sqrt{\frac{2}{A}} \sin\left(\frac{2\pi k x}{A}\right)$$

$$b_k = 2 \sqrt{\frac{2}{A}} \int_0^{A/2} f(x) \sin\left(\frac{2\pi k x}{A}\right) dx.$$

So, if a fcn is given on $[0, \frac{A}{2}]$, say $f \in C[0, \frac{A}{2}]$

$\equiv$ three ways to extend it to a periodic fcn

- $O(1/k)$ (1) Brute force periodization with period $A/2$.
- $O(1/k^2)$ (2) Even extension followed by A -periodization.
- $O(1/k)$ (3) Odd " " "

(2) is the best among these 3 in terms of the decay of the Fourier coeff's.

However, $\equiv$ an even better way!

★ The Lanczos Method (1938)

Suppose $f \in C^{2m}[0, 1]$, but $f(0) \neq f(1)$ ^{no match at $x=0, 1$}
also assume $f^{(2m+1)} \in BV[0, 1]$. $m=1, 2, \dots$

Lanczos's idea:

decompose $f(x) = u(x) + v(x)$

where $u(x)$ = a polynomial of degree $2m-1$.

s.t.
$$\begin{cases} u^{(2k)}(0) = f^{(2k)}(0) \\ u^{(2k)}(1) = f^{(2k)}(1) \end{cases}, \quad k=0, 1, \dots, m-1.$$

Then, consider the **odd** extension of v .

$\Rightarrow v \in C^{2m-1}(\mathbb{R})$, $v^{(k)}(-1) = -v^{(k)}(1)$
 $k=0, 1, 2, \dots, 2m-1.$

and the Fourier sine coefficients of $v(x)$ (with period 2) decay as

$$b_k = O(|k|^{-2m-1})$$

e.g., $m=1$ gives us $b_k = O(1/k^3)$.
and $u(x)$ is a straight line connecting $(0, f(0))$ & $(1, f(1))$.

Remarks:

(1) $f = u + v =$ (an algebraic poly) + (a trig. poly).
can avoid both the **Runge** & **Gibbs** phenomena!

(2) NS-J.F. Remy (2006) generalized this to $\mathbb{R}^d$.