Lecture 9: From the Sturm-Liouville Theory to Discrete Cosine/Sine Transforms

Note Title

☆ Fourier Series, Boundary Value Problems, and the Sturm-Liouville Theory

It is important to note that the Fourier basis fcns $\left\{ \frac{1}{\sqrt{A}} e^{2\pi i k x/A} \right\}_{k \in \mathbb{Z}}$ are eigenfunctions of the following BVP of the 2nd order ODE:

1D Laplacian eigenval. Problem!

$$\begin{cases} -u''(x) = \lambda u(x) & x \in [-\frac{A}{2}, \frac{A}{2}] \\ u(-A/2) = u(A/2) \\ u'(-A/2) = u'(A/2) \end{cases} \Big\} \text{ periodic bdry. cond.!}$$

This is one example of the so-called regular Sturm-Liouville Problem.

λ is the eigenvalue, in fact $\lambda = \lambda_k = (2\pi k/A)^2$, and the corresponding eigenfcn is $\varphi_k(x) = \frac{1}{\sqrt{A}} e^{2\pi i k x/A}$.

<u>Def</u>. A regular Sturm-Liouville problem on the interval $I = [a, b]$ is specified by the following data:

(i) A formally self-adjoint differential operator $\mathcal{L}$ defined as

$$\mathcal{L} u(x) := \frac{1}{w(x)} \left\{ -(p(x) u'(x))' + q(x) u(x) \right\}, \quad \forall x \in I.$$

where $p \in C^1(I)$, $q, w \in C(I)$, $p > 0$, $w > 0$, $q \in \mathbb{R}$, $\forall x \in I$.

(ii) A set of self-adjoint bdry. cond.'s
$B_1(u) = 0$ & $B_2(u) = 0$ for $\mathcal{L}$.

The objective of a regular SL problem is to find all solutions of the following BVP:

$$\begin{cases} \mathcal{L}u = \lambda u \\ B_1(u) = B_2(u) = 0 \end{cases}$$

$\Rightarrow$ Solutions exist for specific λ, i.e., eigenvalues of such rSLP.

Define $L^2_w[a,b] := \{ f \mid \|f\|_{2,w} < \infty \}$
where $\|f\|^2_{2,w} = \|f\|^2_w := \int_a^b |f(x)|^2 w(x)\,dx$

Define the weighted inner product :

$$\langle f, g \rangle_w := \int_a^b f(x)\,\overline{g(x)}\,w(x)\,dx .$$

Going back to the operator $\mathcal{L}$.
$\forall f, g \in L^2_w[a,b]$,

$$\langle \mathcal{L}f, g \rangle_w = \langle f, \mathcal{L}^* g \rangle_w$$

Int. by parts

$$= \langle f, \mathcal{L} g \rangle_w + \Big[-p\,(f'\bar{g} - f\bar{g}') \Big]_a^b$$

if $\mathcal{L}$ is formally self-adjoint.

If $\mathcal{B}_j(f) = \mathcal{B}_j(g) = 0$, $j=1,2$, $f, g \in L^2_w[a,b]$, lead to $\left[-p(f'\bar{g} - f\bar{g}')\right]_a^b = 0$, then these bdy. cond.'s are said to be **self-adjoint**, and together with the formally self-adjoint operator $\mathcal{L}$, we have:

$$\langle \mathcal{L}f, g\rangle_w = \langle f, \mathcal{L}g\rangle_w$$

In this case, the problem is called **self-adjoint**.

The most important thm's here are:

Thm For every rSLP, the following holds:

(a) All eigenvalues are **real**;

(b) Eigenfunctions corresponding to distinct eigenvalues are **orthogonal** w.r.t. $\langle\cdot,\cdot\rangle_w$;

(c) The eigenspace (i.e., the subspace spanned by those eigenfcn's belonging to an eigenval.) for any eigenvalue λ is at most 2 dim. If the bdy. cond. is separated, it is always 1 dim.

Thm For every rSLP, $\exists$ an **ONB** $\{\varphi_n\}_{n\in\mathbb{N}}$ of $L^2_w[a,b]$ s.t. $\{\varphi_n\}$ are **eigenfcn's**. If λ_n : the corresp. eigenval. to φ_n, then $\lambda_n \to \infty$ as $n \to \infty$. Moreover if $f \in C^2[a,b]$, $\mathcal{B}_1(f) = \mathcal{B}_2(f) = 0$, then

$$\sum_{n=1}^{N} \langle f, \varphi_n\rangle_w \varphi_n \to f \text{ **uniformly** as } N \to \infty.$$

So, $\begin{cases} -u'' = \lambda u, \\ u(-A/2) = u(A/2) \\ u'(-A/2) = u'(A/2) \end{cases}$ is one of the simplest rSLPs!

Remark: A singular SLP is an SLP with
(i) $p(a) = 0$ or $p(b) = 0$.
in addition $w(a) = 0$ or $+\infty$ or $w(b) = 0$ or $+\infty$;
or (ii) $a = -\infty$ or $b = +\infty$.

Almost all of classical orthogonal polynomials are generated by sSLP's with specific B.C. & weight fcn's.

	p	q	w	λ_n	$[a,b]$
Legendre poly.	$1-x^2$	0	1	$n(n+1)$	$[-1,1]$
Chebyshev poly.	$\sqrt{1-x^2}$	0	$1/\sqrt{1-x^2}$	n^2	$[-1,1]$
Hermite poly.	e^{-x^2}	0	e^{-x^2}	$2n$	$(-\infty,\infty)$
$\alpha>-1$ Laguerre poly.	$x^{\alpha+1}e^{-x}$	0	$x^{\alpha}e^{-x}$	n	$[0,\infty)$
Prolate Spheroidal wave fcn's	$1-x^2$	c^2x^2	1	λ $(0<\lambda<1)$	$[-1,1]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

In 2D and higher, the problem becomes more intricate, of course.
The simplest version is that of Laplacian eigenvalue problem:

$$-\Delta u = \lambda u \quad \text{in } \Omega = \text{a domain in } \mathbb{R}^d$$

Dirichlet / Neumann $\left\{ \begin{array}{l} u = 0 \\ \text{or } \partial_\nu u = 0 \end{array} \right.$ on $\partial\Omega$ (a bdry of Ω).

☆ Fourier Sine & Cosine Series

... come out naturally as eigenfcn's on the simple r SLP with the following B.C.'s:

$$-u'' = \lambda u, \quad x \in \left[-\frac{A}{2}, \frac{A}{2}\right].$$

i.e., $p(x) \equiv 1$, $q(x) \equiv 0$, $w(x) \equiv 1$.

Dirichlet B.C.: $u\left(-\frac{A}{2}\right) = u\left(\frac{A}{2}\right) = 0 \Rightarrow \left\{\sqrt{\frac{2}{A}} \sin \frac{2\pi k}{A} x\right\}_{k=1}^{\infty}$

Neumann B.C.: $u'\left(-\frac{A}{2}\right) = u'\left(\frac{A}{2}\right) = 0 \Rightarrow \left\{\frac{1}{\sqrt{A}}\right\} \cup \left\{\sqrt{\frac{2}{A}} \cos \frac{2\pi k}{A} x\right\}_{k=1}^{\infty}$

But clearly $\exists$ other possibilities, e.g.,

$$u\left(-\frac{A}{2}\right) = u'\left(\frac{A}{2}\right) = 0 \Rightarrow \left\{\sqrt{\frac{2}{A}} \sin \frac{2\pi\left(k+\frac{1}{2}\right)}{A} x\right\}_{k=0}^{\infty}$$

$$u'\left(-\frac{A}{2}\right) = u\left(\frac{A}{2}\right) = 0 \Rightarrow \left\{\sqrt{\frac{2}{A}} \cos \frac{2\pi\left(k+\frac{1}{2}\right)}{A} x\right\}_{k=0}^{\infty}$$

Discretization gives us further intricacies!

☆ Discrete Sine & Cosine Transforms

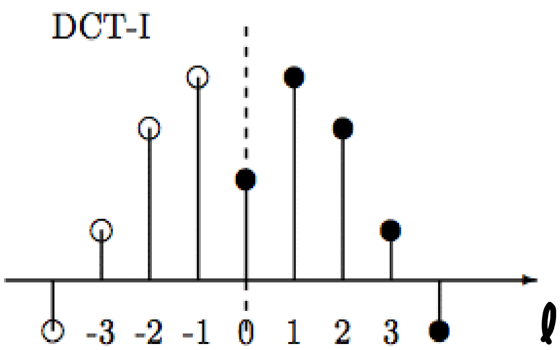
not $\frac{2\pi k \ell}{N}$ Type

Define

$$\begin{cases} F_s^{I}[k] := \sum_{\ell=1}^{N-1} f[\ell] \sin \frac{\pi k \ell}{N} & : \text{DST-I} \\ F_c^{I}[k] := \sum_{\ell=0}^{N-1} f[\ell] \cos \frac{\pi k \ell}{N} & : \text{DCT-I} \end{cases}$$

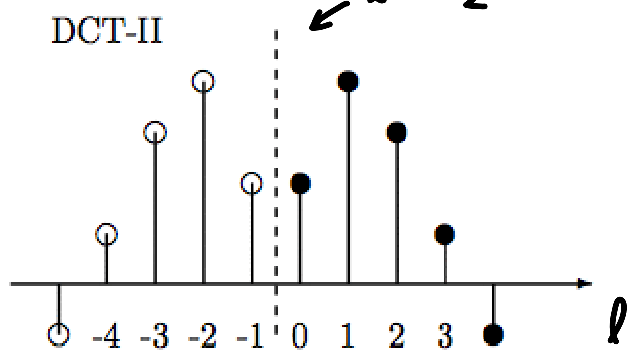
These can be computed via normal FFT (of length $2N$) by appropriately extending the original sequence.

$\exists$ 4 different types of DSTs & DCTs with different B.C.'s.



symmetry at grid point

$\ell = -\frac{1}{2}$



symmetry at mid point

For convenience, let's define the following weight vector for orthonormality:

$$b[\ell] := \begin{cases} 0 & \text{if } \ell < 0 \text{ or } \ell > N; \\ 1/\sqrt{2} & \text{if } \ell = 0 \text{ or } \ell = N; \\ 1 & \text{if } 1 \leq \ell \leq N-1. \end{cases}$$

Now we can define the following transf. matrices:

$$\begin{cases} \text{DCT-I}: C_{N+1}^{\mathrm{I}} \in \mathbb{R}^{(N+1)\times(N+1)}, & C_{N+1}^{\mathrm{I}}[k,\ell] = b[k]\,b[\ell]\sqrt{\frac{2}{N}}\cos\frac{\pi k \ell}{N} \\ \text{DCT-II}: C_N^{\mathrm{II}} \in \mathbb{R}^{N\times N}, & C_N^{\mathrm{II}}[k,\ell] = b[k]\sqrt{\frac{2}{N}}\cos\frac{\pi k(\ell+\frac{1}{2})}{N} \\ \text{DCT-III}: C_N^{\mathrm{III}} \in \mathbb{R}^{N\times N}, & C_N^{\mathrm{III}}[k,\ell] = b[\ell]\sqrt{\frac{2}{N}}\cos\frac{\pi (k+\frac{1}{2})\ell}{N} \\ \text{DCT-IV}: C_N^{\mathrm{IV}} \in \mathbb{R}^{N\times N}, & C_N^{\mathrm{IV}}[k,\ell] = \sqrt{\frac{2}{N}}\cos\frac{\pi (k+\frac{1}{2})(\ell+\frac{1}{2})}{N} \end{cases}$$

For DCT-I, $k, \ell = 0, 1, \cdots, N$

For others, $k, \ell = 0, 1, \cdots, N-1$.

k: frequency index

ℓ: space (or time) index.

$$
\begin{cases}
\text{DST-I}: S_{N-1}^{\mathrm{I}} \in \mathbb{R}^{(N-1)\times(N-1)}, & S_{N-1}^{\mathrm{I}}[k,l] = \sqrt{\frac{2}{N}} \sin \frac{\pi k l}{N} \\
\text{DST-II}: S_N^{\mathrm{II}} \in \mathbb{R}^{N\times N}, & S_N^{\mathrm{II}}[k,l] = b[k+1] \sqrt{\frac{2}{N}} \sin \frac{\pi (k+1)(l+\frac{1}{2})}{N} \\
\text{DST-III}: S_N^{\mathrm{III}} \in \mathbb{R}^{N\times N}, & S_N^{\mathrm{III}}[k,l] = b[l+1] \sqrt{\frac{2}{N}} \sin \frac{\pi (k+\frac{1}{2})(l+1)}{N} \\
\text{DST-IV}: S_N^{\mathrm{IV}} \in \mathbb{R}^{N\times N}, & S_N^{\mathrm{IV}}[k,l] = \sqrt{\frac{2}{N}} \sin \frac{\pi (k+\frac{1}{2})(l+\frac{1}{2})}{N}
\end{cases}
$$

For DST-I, $k, l = 1, \cdots, N-1$
For others, $k, l = 0, 1, \cdots, N-1$.

Remarks:

(1) In the JPEG image compression standard, the 2D version of DCT-II is used on patches of size 8×8 pixels via the tensor product of 1D DCT-II.

(2) "dct" in both Julia and MATLAB is the DCT-II, and is included in FFTW.jl and the Signal Processing Toolbox, respectively.

(3) The MATLAB function "dst" is the DST-I (unnormalized version), and is included in the PDE Toolbox. In Julia/FFTW.jl, it is not explicitly defined. However, you can run

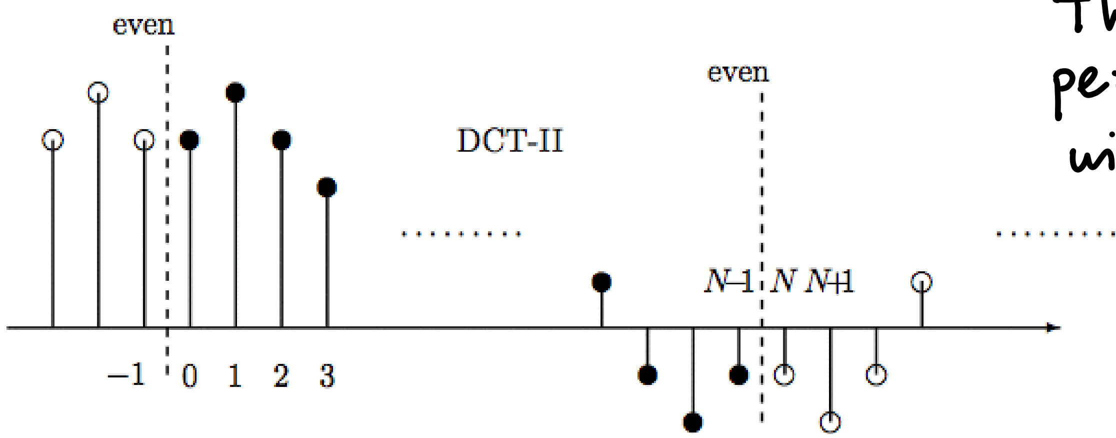
```
julia> y = FFTW.r2r(x, FFTW.RODFT00)
```

for DST-I of a vector x. However, it's not unitary, so you may want to normalize "y".

★ Comments on the B.C.'s

	left endpt.	right endpt
DCT-I	grid pt, Neumann (-1, 0, 1)	grid pt, Neumann ($N-1$, N, $N+1$)
II	mid pt, Neumann ($-\frac{1}{2}$)	mid pt, Neumann ($N-\frac{1}{2}$)
III	grid pt, Neumann (-1, 0, 1)	grid pt, Dirichlet ($N-1$, N, $N+1$)
IV	mid pt, Neumann ($-\frac{1}{2}$, 0, 1)	mid pt, Dirichlet ($N-\frac{1}{2}$)
DST-I	grid pt, Dirichlet (-1, 0, 1)	grid pt, Dirichlet ($N-1$, N, $N+1$)
II	mid pt, Dirichlet (-1, $-\frac{1}{2}$, 0)	mid pt, Dirichlet ($N-\frac{1}{2}$)
III	grid pt, Dirichlet (-1, 0, 1)	grid pt, Neumann ($N-1$, N, $N+1$)
IV	mid pt, Dirichlet (-1, $-\frac{1}{2}$, 0)	mid pt, Neumann ($N-\frac{1}{2}$)

★ DCT-II



Then periodized with period 2N

Define

$$\tilde{f}[\ell] := \begin{cases} f[\ell] & \text{if } \ell = 0, 1, \cdots, N-1. \\ f[2N-\ell-1] & \text{if } \ell = N, \cdots, 2N-1. \end{cases}$$

Then consider

$$\mathcal{D}_{2N}\{\tilde{f}\}[k] = \sum_{\ell=0}^{2N-1} \tilde{f}[\ell]\, \omega_{2N}^{-k\ell}$$

$$= \sum_{\ell=0}^{N-1} f[\ell]\, \omega_{2N}^{-k\ell} + \sum_{\ell=N}^{2N-1} f[2N-\ell-1]\, \omega_{2N}^{-k\ell}$$

$$= \sum_{\ell=0}^{N-1} f[\ell]\, \omega_{2N}^{-k\ell} + \sum_{m=N-1}^{0} f[m]\, \omega_{2N}^{-k(2N-1-m)} \quad \leftarrow 2N-\ell-1 = m$$

$$= \sum_{\ell=0}^{N-1} f[\ell]\, \omega_{2N}^{-k\ell} + \sum_{m=0}^{N-1} f[m]\, \omega_{2N}^{km} \cdot \omega_{2N}^{k} \quad \leftarrow \omega_{2N}^{-k\cdot 2N} = 1$$

$$= \sum_{\ell=0}^{N-1} f[\ell] \left(\omega_{2N}^{-k\ell} + \omega_{2N}^{(\frac{k}{2}+\frac{k}{2})} \cdot \omega_{2N}^{k\ell} \right)$$

$$= \omega_{2N}^{k/2} \sum_{\ell=0}^{N-1} f[\ell] \left(\omega_{2N}^{-k(\ell+\frac{1}{2})} + \omega_{2N}^{k(\ell+\frac{1}{2})} \right)$$

$$= 2 e^{\frac{\pi i k}{N}} \sum_{\ell=0}^{N-1} f[\ell] \cos \frac{\pi k (\ell+\frac{1}{2})}{N}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2N}} \mathcal{D}_{2N}\{\tilde{f}\}[k] = \sqrt{\frac{2}{N}}\, e^{\frac{\pi i k}{N}} \sum_{\ell=0}^{N-1} f[\ell] \cos \frac{\pi k(\ell+\frac{1}{2})}{N}$$

Viewing samples at half integers on the x-axis in the DFT set up eliminates this phase factor.

- The inverse transform to DCT-II is DCT-III !