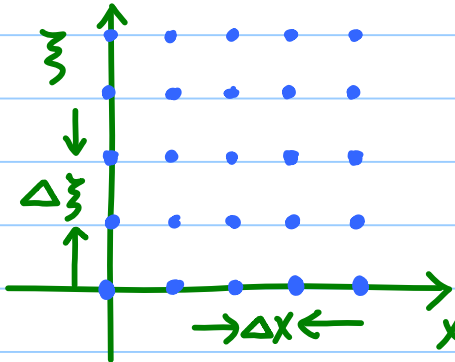


Intro to Frame Theory

Lecture 12: The Balian-Low Theorem

Note Title

★ Sampling & WFT



Consider sampling $f \in L^2(\mathbb{R})$ on the time-frequency plane via WF-atoms of the form:

$$g_{m,n}(x) := g(x - m\Delta x) e^{2\pi i n \Delta \xi x}, \quad m, n \in \mathbb{Z}$$

instead of $g_{x_0, \xi_0}(x) = g(x - x_0) e^{2\pi i \xi_0 x}$. This is Gabor's proposal (1946).

$$f(x) = \sum \alpha_{m,n} \tilde{g}_{m,n}(x), \quad \alpha_{m,n} = \langle f, g_{m,n} \rangle$$

Synthesis dual basis (or dual frame) Analysis

$\{g_{m,n}\}_{(m,n) \in \mathbb{Z}^2}$ may be quite redundant (i.e., more than a basis), and may not be orthogonal.

These ideas lead to:

★ The Frame Theory

Def. Let \mathcal{H} be a Hilbert space with its norm $\|\cdot\|$ and let $\{\phi_\gamma\}_{\gamma \in \Gamma} \subset \mathcal{H}$.

Then $\{\phi_\gamma\}$ is said to constitute a **frame** of \mathcal{H} if $\forall f \in \mathcal{H}, \exists A, B \neq 0$ s.t.

$$(*) \quad A \|f\|^2 \leq \sum_{\gamma \in \Gamma} |\langle f, \phi_\gamma \rangle|^2 \leq B \|f\|^2$$

where the constants A, B are referred to as the **frame bounds**.

Remarks:

(1) Existence of the frame bounds $B \geq A > 0$ is a necessary & sufficient condition for the invertibility of the frame operator, i.e., you can reconstruct your original fcn f from the frame coefficients $\{\langle f, \phi_\gamma \rangle\}_{\gamma \in \Gamma}$.

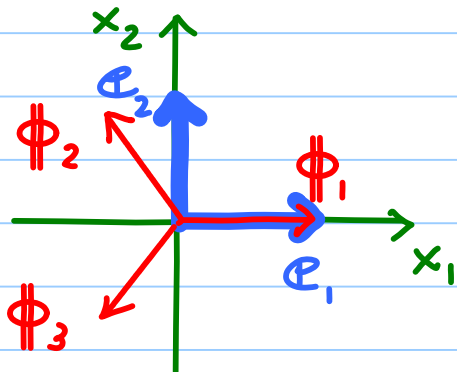
(2) If $\|\phi_\gamma\| = 1, \forall \gamma \in \Gamma$, and $A > 1$, then the frame is **redundant**, and this A can be interpreted as a **minimum redundancy factor**.

(3) If $\|\phi_\gamma\| = 1, \forall \gamma \in \Gamma$, and $\{\phi_\gamma\}_{\gamma \in \Gamma}$ are linearly independent, then $A \leq 1 \leq B$. In this case, $\{\phi_\gamma\}_{\gamma \in \Gamma}$ is **not** redundant and forms a **basis** of \mathcal{H} , which is called a **Riesz basis**.

(4) $A = B = 1$ \Leftrightarrow $\{\phi_\gamma\}_{\gamma \in \Gamma}$: an **ONB** of \mathcal{H} .
 \Rightarrow a **Parseval frame** \exists a Parseval frame that is not an ONB. If we assume $\{\phi_\gamma\}$: a Riesz basis, then $A = B = 1$ implies $\{\phi_\gamma\}$: an ONB.

(5) In the case of $A = B > 1$, $\{\phi_\gamma\}_{\gamma \in \Gamma}$ is called a **tight** frame, and is redundant. Its redundancy is measured by A .

A simple example in \mathbb{R}^2



$$\begin{cases} \phi_1 = e_1 \\ \phi_2 = \frac{-1}{2}e_1 + \frac{\sqrt{3}}{2}e_2 \\ \phi_3 = \frac{-1}{2}e_1 - \frac{\sqrt{3}}{2}e_2 \end{cases}$$

$\{\phi_1, \phi_2, \phi_3\}$ form a frame of \mathbb{R}^2 .

Let $\forall f \in \mathbb{R}^2$, $f = f_1 e_1 + f_2 e_2$, $f_1, f_2 \in \mathbb{R}$.

Then $\langle f, \phi_1 \rangle = f_1$, $\langle f, \phi_2 \rangle = -\frac{1}{2}f_1 + \frac{\sqrt{3}}{2}f_2$

$$\langle f, \phi_3 \rangle = -\frac{1}{2}f_1 - \frac{\sqrt{3}}{2}f_2$$

$$\begin{aligned} \Rightarrow \sum_{\gamma=1}^3 |\langle f, \phi_\gamma \rangle|^2 &= f_1^2 + \left(-\frac{1}{2}f_1 + \frac{\sqrt{3}}{2}f_2\right)^2 + \left(-\frac{1}{2}f_1 - \frac{\sqrt{3}}{2}f_2\right)^2 \\ &= \frac{3}{2}(f_1^2 + f_2^2) = \frac{3}{2} \|f\|^2 \end{aligned}$$

$\Rightarrow A = B = \frac{3}{2}$, i.e., it's a tight frame

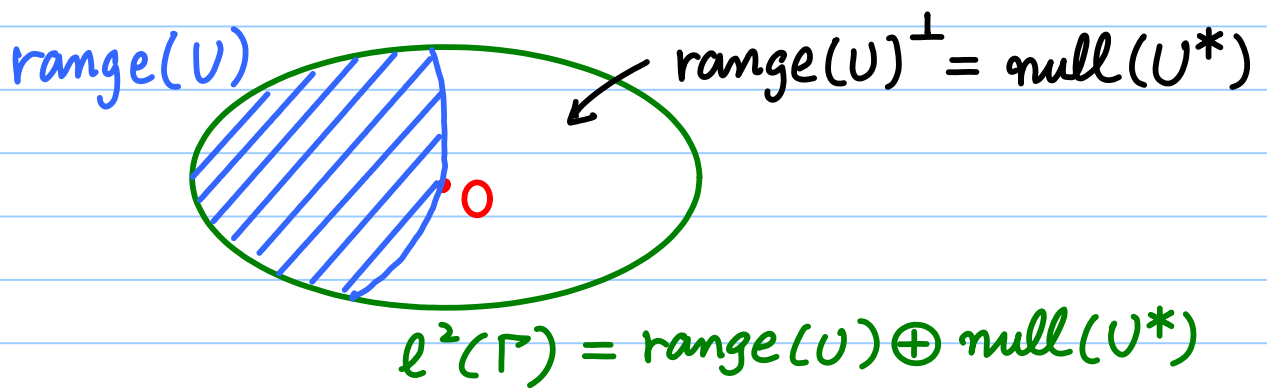
\hookrightarrow with 1.5 redundancy (agrees with our intuition!)
 $\left\{ \frac{\sqrt{2}}{3}\phi_1, \frac{\sqrt{2}}{3}\phi_2, \frac{\sqrt{2}}{3}\phi_3 \right\}$ form a Parseval frame.

★ Frame Operator and Reconstruction

Def. $U: \mathcal{H} \rightarrow \ell^2(\Gamma)$ is called a **frame operator** if $Uf[\gamma] = \langle f, \phi_\gamma \rangle$ for $f \in \mathcal{H}$.

Note that Uf is a sequence indexed by $\gamma \in \Gamma$.

Prop. If $\{\phi_\gamma\}_{\gamma \in \Gamma}$ is a frame and linearly **dependent**, then $\text{range}(U) \subsetneq \ell^2(\Gamma)$.



(Pf) $\forall f \in \mathfrak{H}$,

$$\|Uf\|^2 = \sum_{\gamma \in \Gamma} |\langle f, \phi_\gamma \rangle|^2 \leq B \|f\|^2 < \infty$$

So clearly $Uf \in l^2(\Gamma)$ so $\text{range}(U) \subset l^2(\Gamma)$.
 But $\{\phi_\gamma\}_{\gamma \in \Gamma}$ is linearly dependent, so
 $\exists c \in l^2(\Gamma), c \neq 0$ s.t. $\sum_{\gamma \in \Gamma} c[\gamma] \phi_\gamma(x) = 0$.

Then for any $f \in \mathfrak{H}$,

$$\begin{aligned} \sum_{\gamma \in \Gamma} \bar{c}[\gamma] Uf[\gamma] &= \sum_{\gamma \in \Gamma} \bar{c}[\gamma] \langle f, \phi_\gamma \rangle \\ &= \sum_{\gamma \in \Gamma} \langle f, c[\gamma] \phi_\gamma \rangle = \langle f, \underbrace{\sum_{\gamma \in \Gamma} c[\gamma] \phi_\gamma}_{=0} \rangle = 0 \end{aligned}$$

$\Rightarrow Uf \perp c \neq 0$, i.e., $\text{range}(U) \perp c \neq 0$.

$\Rightarrow c \in \text{range}(U)^\perp = \text{null}(U^*)$

and $\text{range}(U) \subsetneq l^2(\Gamma)$. ///

Thm The frame operator U has a **pseudo inverse** $U^\dagger = (U^* U)^{-1} U^*$

s.t. $\|U^\dagger\| \leq \frac{1}{\sqrt{A}}$. $\xrightarrow{\text{operator norm}} = \sup_{\substack{\alpha \in l^2(\Gamma) \\ \alpha \neq 0}} \frac{\|U^\dagger \alpha\|_{\mathfrak{H}}}{\|\alpha\|_{l^2(\Gamma)}}$

Thm Let $\{\phi_\gamma\}_{\gamma \in \Gamma}$ be a frame of \mathcal{H} with its frame bounds A, B .

Define $\tilde{\phi}_\gamma := (U^*U)^{-1} \phi_\gamma$, $\gamma \in \Gamma$

Then $\forall f \in \mathcal{H}$,

$$\frac{1}{B} \|f\|^2 \leq \sum_{\gamma \in \Gamma} |\langle f, \tilde{\phi}_\gamma \rangle|^2 \leq \frac{1}{A} \|f\|^2$$

$$\begin{aligned} \text{and } f &= U^* U f = \sum_{\gamma \in \Gamma} \langle f, \phi_\gamma \rangle \tilde{\phi}_\gamma \\ &= \sum_{\gamma \in \Gamma} \langle f, \tilde{\phi}_\gamma \rangle \phi_\gamma \end{aligned}$$

If $A = B$ (i.e., tight), then $\tilde{\phi}_\gamma = \frac{1}{A} \phi_\gamma$

$A = B = 1, \|\phi_\gamma\| = 1, \forall \gamma \in \Gamma \Leftrightarrow \tilde{\phi}_\gamma = \phi_\gamma, \{\phi_\gamma\}$: an ONB of \mathcal{H} .

The system $\{\tilde{\phi}_\gamma\}_{\gamma \in \Gamma}$ is called the **dual frame** of \mathcal{H} relative to $\{\phi_\gamma\}_{\gamma \in \Gamma}$.

(Pf) Not difficult if we define the dual frame operator $\tilde{U} := U(U^*U)^{-1}$, and assume the following lemma:

Lemma Let \mathcal{L} be a self-adjoint operator in \mathcal{H} s.t. $\forall f \in \mathcal{H}, \exists B \geq A > 0$,

$$A \|f\|^2 \leq \langle \mathcal{L}f, f \rangle \leq B \|f\|^2.$$

Then \mathcal{L} is invertible and

$$\frac{1}{B} \|f\|^2 \leq \langle \mathcal{L}^{-1}f, f \rangle \leq \frac{1}{A} \|f\|^2.$$

See Mallat's book (sec. 5.1.2 for the detail). ///

Let's go back to the WFT - Gabor proposal.

Thm (Daubechies, 1990) [The Necessary Cond.]

The WF family $\{g_{m,n}(x) = g(x - m\Delta x) e^{2\pi i n \Delta \xi x}\}_{(m,n) \in \mathbb{Z}^2}$

constitute a frame of $L^2(\mathbb{R})$

only if

$$\Delta x \Delta \xi \leq 1$$



The frame bounds A, B necessarily satisfy

$$(*) \quad A \leq \frac{1}{\Delta x \Delta \xi} \leq B$$

$$\left\{ \begin{array}{l} A \leq \frac{1}{\Delta \xi} \sum_{m \in \mathbb{Z}} |g(x - m\Delta x)|^2 \leq B \quad \forall x \in \mathbb{R} \\ A \leq \frac{1}{\Delta x} \sum_{n \in \mathbb{Z}} |\hat{g}(\xi - n\Delta \xi)|^2 \leq B \quad \forall \xi \in \mathbb{R} \end{array} \right.$$

\rightarrow no gaps on the x and ξ axes.

Remarks:

(1) If we want to make $\{g_{m,n}\}_{(m,n) \in \mathbb{Z}^2}$ an ONB of $L^2(\mathbb{R})$, then we must have $\Delta x \Delta \xi = 1$ (critical sampling) since $A = B = 1$ together with $(*)$ forces this.

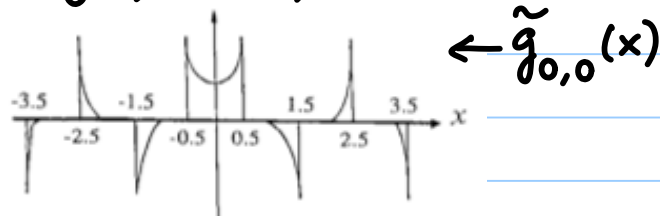
necessary but not sufficient

(2) In 1980, M. Baostians tried to compute the dual of the Gabor frame with $\Delta x \Delta \xi = 1$.

g is Gaussian

He got very singular $\{\tilde{g}_{m,n}\}_{(m,n) \in \mathbb{Z}^2}$

e.g., $\sigma_{\xi}(\tilde{g}_{m,n}) = +\infty$



(3) In early 1980's, Roger Balian tried to construct an ONB of the form $\{g_{m,n}\}_{(m,n) \in \mathbb{Z}^2}$ for a general window fcn g (not necessarily Gaussian) with $\|g\| = 1$. Then he proved that if $\{g_{m,n}\}_{(m,n) \in \mathbb{Z}^2}$ form an ONB of $L^2(\mathbb{R})$, then $\Delta x \Delta \xi = 1$ without using the frame theory.

Examples of such $g = g_{0,0}$

- $g(x) = \chi_{[-\frac{1}{2}, \frac{1}{2}]}(x) \rightarrow$ generates block w.f. transform

But, $\sigma_{\xi}(g_{m,n}) = +\infty$ as shown before.

- $g(x) = \text{sinc}(x) \rightarrow$ generates the transf.

But, $\sigma_x(g_{m,n}) = +\infty$ by sharp partitioning of the frequency domain.

Is this an accident?

No! We have the following

Thm (Balian-Low, 1981, 1985)

Suppose $\{g_{m,n}\}_{(m,n) \in \mathbb{Z}^2}$ constitutes a windowed Fourier frame of $L^2(\mathbb{R})$ with $\Delta x \Delta \xi = 1$ (which includes the case of an ONB).

Then either $\sigma_x(g) = +\infty$ or $\sigma_{\xi}(g) = +\infty$.

(Pf) We only prove here the ONB case due to Guy Battle (1988). For the general non-orthogonal case including the Gabor frame case, see Daubechies & Janssen (1993).

Our strategy: Assume $\sigma_x(g) < \infty$ and $\sigma_{\xi}(g) < \infty$. Then these lead to contradiction.

Consider $\langle xg, g' \rangle$, which also appeared in the proof of the Heisenberg inequality.

Note $xg, g' \in L^2(\mathbb{R})$ because

$$\|xg(x)\|^2 = \int x^2 |g(x)|^2 dx = \sigma_x^2(g) < +\infty$$

since $m_x(g) = 0$ and $\|g\| = 1$

Similarly, $\hat{g}' = 2\pi i \xi \hat{g}$ & $\sigma_\xi(g) < +\infty$

lead to $\|g'\| < \infty$, i.e., $g' \in L^2(\mathbb{R})$.

$$\begin{aligned} \text{Now, } \langle xg, g' \rangle &= \sum_m \sum_n \langle xg, g_{m,n} \rangle \langle g_{m,n}, g' \rangle \\ &\stackrel{(a)}{=} \sum_m \sum_n \langle g_{-m,-n}, xg \rangle \langle -(g')_{m,n}, g \rangle \\ &\stackrel{(b)}{=} \sum_m \sum_n \langle g_{-m,-n}, xg \rangle \langle -g', g_{-m,-n} \rangle \\ &= \sum_m \sum_n \langle -g', g_{m,n} \rangle \langle g_{m,n}, xg \rangle \\ &\stackrel{\{g_{m,n}\}: \text{ONB}}{=} -\langle g', xg \rangle \quad (1) \end{aligned}$$

We'll show the justification (a), (b) later.

Now, Consider a function $f \in C_c^\infty(\mathbb{R})$, i.e., a space of C^∞ functions vanishing as $|x| \rightarrow \infty$.

$$\begin{aligned} \text{Then, } \langle xf, f' \rangle &= \int_{-\infty}^{\infty} xf(x) \overline{f'(x)} dx \\ &= \cancel{xf(x) \overline{f(x)}} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \overline{f(x)} (xf'(x) + f(x)) dx \\ &= -\int_{-\infty}^{\infty} x \overline{f(x)} f'(x) dx - \|f\|^2 \\ &= -\langle f', xf \rangle - \|f\|^2 \end{aligned}$$

Since $C_c^\infty(\mathbb{R})$ is dense in $\mathcal{H} = \{f \in L^2 \mid xf, f' \in L^2\}$, the window for g under consideration must satisfy $\langle xg, g' \rangle = -\langle g', xg \rangle - \|g\|^2$ (2)

Combining (1) & (2), we conclude $\|g\| = 0$, which contradicts with $\|g\| = 1 \neq 0$

Finally, the justification of (a):

$$\langle xg, g_{m,n} \rangle = \langle g_{-m,-n}, xg \rangle \quad \& \quad \langle g_{m,n}, g' \rangle = \langle -(g')_{m,n}, g \rangle$$

$$\begin{aligned} \textcircled{\ominus} \quad \langle xg, g_{m,n} \rangle &= \int xg(x) \overline{g(x-m\Delta x)} e^{-2\pi i n \Delta \xi x} dx \\ &\stackrel{y=x-m\Delta x}{=} \int (y+m\Delta x) g(y+m\Delta x) \overline{g(y)} e^{-2\pi i n \Delta \xi y} \cdot e^{-2\pi i n \Delta \xi m \Delta x} dy \\ &= \int (x+m\Delta x) g(x+m\Delta x) \overline{g(x)} e^{-2\pi i n \Delta \xi x} dx \quad \text{since } \Delta x \Delta \xi = 1. \end{aligned}$$

$$= \langle g_{-m,-n}, xg \rangle + m\Delta x \langle g_{-m,-n}, g \rangle$$

$$= \langle g_{-m,-n}, xg \rangle \quad \parallel$$

$= 0$ since $g = g_{0,0}$.
 $\{g_{m,n}\}$: ONB. so,
 $(m,n) \neq (0,0)$. $g_{-m,-n} \perp g$
 if $(m,n) = (0,0)$. $m\Delta x = 0$.

$$\langle g_{m,n}, g' \rangle = \int_{-\infty}^{\infty} g_{m,n}(x) \overline{g'(x)} dx$$

$$= g_{m,n}(x) \overline{g(x)} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (g_{m,n}(x))' \overline{g(x)} dx$$

$$= \langle -(g')_{m,n}, g \rangle - 2\pi i n \Delta \xi \langle g_{m,n}, g \rangle$$

$$= \langle -(g')_{m,n}, g \rangle \quad \parallel \quad \text{the same logic} \quad = 0$$

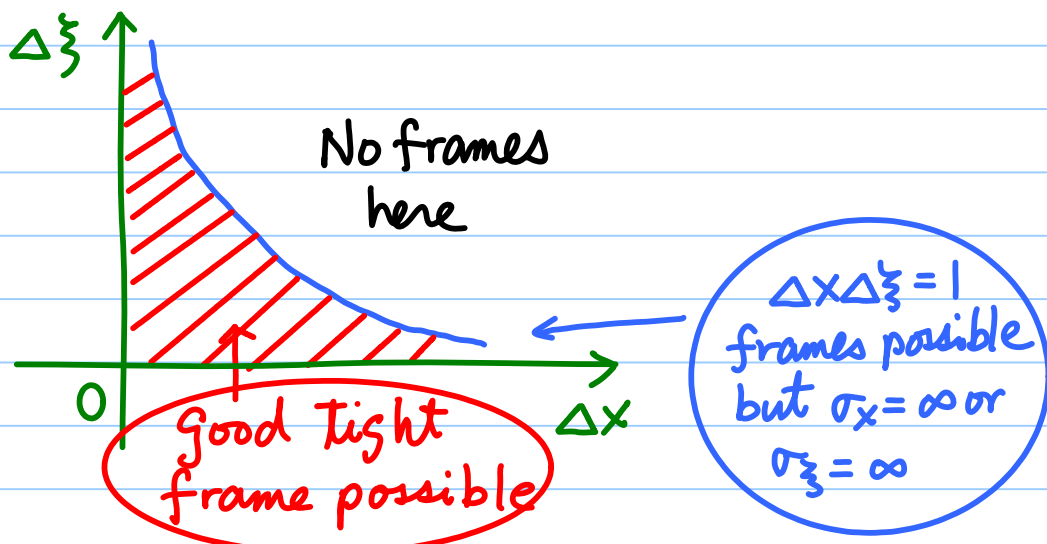
How about the justification (b)?

$$\langle -(g')_{m,n}, g \rangle = \langle -g', g_{-m,-n} \rangle$$

Use the same logic as the first part of (a). //

Continuation of Remarks

(4) In 1990, Daubechies summarized the frame conditions for $\{g_{m,n}\}$ w.r.t. Δx & $\Delta \xi$ as follows:



$\Delta x \Delta \xi \leq 1$ was the necessary cond. for $\{g_{m,n}\}$ to form a frame. How about the **sufficient** cond.?

Thm. (Daubechies, 1990)

Let $\beta(u) := \sup_{0 \leq x \leq \Delta x} \sum_{m \in \mathbb{Z}} |g(x - m\Delta x)| |g(x - m\Delta x + u)|$

$$\Delta := \sum_{k \in \mathbb{Z}} \left[\beta\left(\frac{k}{\Delta \xi}\right) \beta\left(\frac{-k}{\Delta \xi}\right) \right]^{\frac{1}{2}}$$

If $\Delta x, \Delta \xi$ satisfy

$$A_0 := \frac{1}{\Delta \xi} \left(\inf_{0 \leq x \leq \Delta x} \sum_{m \in \mathbb{Z}} |g(x - m\Delta x)|^2 - \Delta \right) > 0$$

$$B_0 := \frac{1}{\Delta \xi} \left(\sup_{0 \leq x \leq \Delta x} \sum_{m \in \mathbb{Z}} |g(x - m\Delta x)|^2 + \Delta \right) < \infty,$$

then $\{g_{m,n}\}_{(m,n) \in \mathbb{Z}^2}$ is a frame with $\begin{cases} A_0 = \inf A \\ B_0 = \sup B \end{cases}$.

See the original paper for the proof.