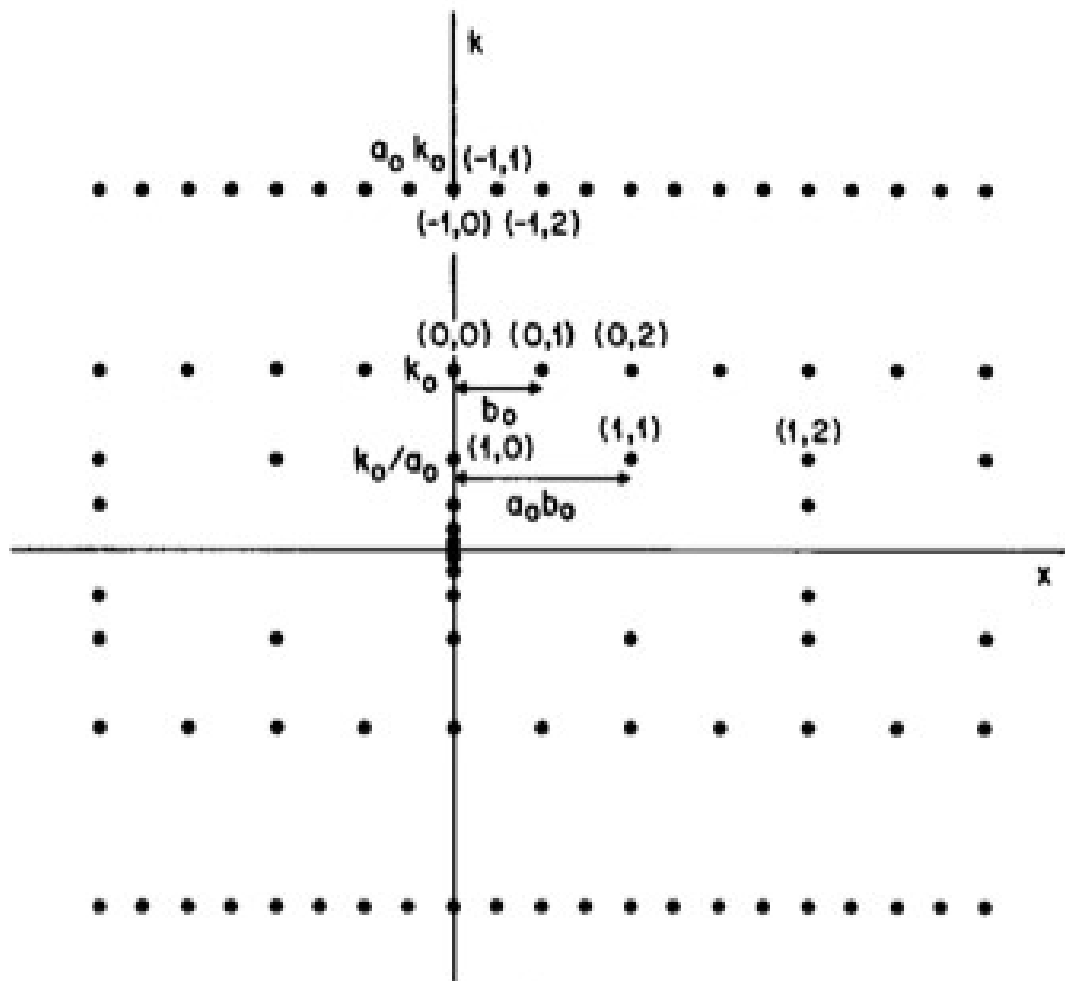


Lecture 15: Wavelet Bases

Note Title

- Recall the sampling lattice pattern ("regular hyperbolic grid") to generate the discrete wavelet transf. from the continuous wavelet transf. $Wf(a, b)$.

$$(a, b) = (a_0^m, n a_0^m b_0), \quad m, n \in \mathbb{Z}$$
$$\psi_{m,n}(x) := a_0^{-m/2} \psi(a_0^{-m} x - n b_0), \quad m, n \in \mathbb{Z}$$
$$= \frac{1}{\sqrt{a_0^m}} \psi\left(\frac{x - n a_0^m b_0}{a_0^m}\right)$$



The most common/popular choice of (a, b) is $a = 2^j, b = 2^j k$ i.e., $a_0 = 2, b_0 = 1$.

$$\Rightarrow \psi_{j,k}(x) = \frac{1}{\sqrt{2^j}} \psi\left(\frac{x - 2^j k}{2^j}\right) = 2^{-j/2} \psi(2^{-j} x - k), \quad (j, k) \in \mathbb{Z}^2$$

View the approximation of f at resolution 2^{-j} as an **orthogonal projection** P_j (or P_{V_j}) of f onto a subspace:

$$V_j := \overline{\text{span}\{\phi_{j,k}, k \in \mathbb{Z}\}} \subset L^2(\mathbb{R})$$

Def. (Multiresolution Approximation)

a sequence of closed subspaces $\{V_j\}_{j \in \mathbb{Z}}$ of $L^2(\mathbb{R})$ is called a **multiresolution approximation (MRA)** if the following six properties are satisfied:

$$(1) \forall (j, k) \in \mathbb{Z}^2, f(x) \in V_j \iff f(x - 2^j k) \in V_j$$

$j \downarrow$: high res.

$$(2) \forall j \in \mathbb{Z}, V_{j+1} \subset V_j$$

$j \uparrow$: low res.

$$(3) \forall j \in \mathbb{Z}, f(x) \in V_j \iff f\left(\frac{x}{2}\right) \in V_{j+1}$$

$$(4) \lim_{j \rightarrow \infty} V_j = \bigcap_{j=-\infty}^{\infty} V_j = \{0\}$$

$$(5) \lim_{j \rightarrow -\infty} V_j = \bigcup_{j=-\infty}^{\infty} V_j = L^2(\mathbb{R})$$

(6) $\exists \theta \in V_0$, s.t. $\{\theta(x-k)\}_{k \in \mathbb{Z}}$ form a Riesz basis for V_0 .

linearly indep. frame (i.e., a basis)
with $0 < A \leq 1 \leq B$ (if $\|\theta\| = 1$).

Remark:

$$P_{V_j} f \in V_j, \quad \underbrace{P_{V_j}^2}_{\text{proj}} = P_{V_j}, \quad \underbrace{P_{V_j}^*}_{\text{orthogonal proj}} = P_{V_j}$$

$$\lim_{j \rightarrow \infty} \|P_{V_j} f\| = 0, \quad \lim_{j \rightarrow -\infty} \|f - P_{V_j} f\| = 0.$$

• Now $\{\theta(x-k)\}_{k \in \mathbb{Z}}$ is a Riesz basis for V_0
 $\iff \forall f \in V_0, \exists! \{a_k\} \in \ell^2(\mathbb{Z})$ s.t.

$$f(x) = \sum_{k \in \mathbb{Z}} a_k \theta(x-k) \text{ with}$$

$$A \|f\|^2 \leq \sum |a_k|^2 \leq B \|f\|^2, \begin{matrix} \exists A \neq 0 \\ \exists B > 0 \end{matrix}$$

Then, we can adapt this $\{\theta(x-k)\}_{k \in \mathbb{Z}}$
 by dilation for V_j , i.e.,
 $\{2^{-j/2} \theta(2^{-j}x - k)\}_{k \in \mathbb{Z}}$ is a Riesz basis for V_j
 with the same frame bounds A, B .

Prop. A family $\{\theta(x-k)\}_{k \in \mathbb{Z}}$ is a Riesz basis
 for $V_0 = \text{span}\{\theta(x-k)\}_{k \in \mathbb{Z}}$ $\iff \exists B \geq A > 0$ s.t.
 $\forall \xi \in [-\frac{1}{2}, \frac{1}{2}], \frac{1}{B} \leq \sum_{k \in \mathbb{Z}} |\hat{\theta}(\xi - k)|^2 \leq \frac{1}{A}$

(Pf) (\Leftarrow) Take any $f \in V_0$.

$$f(x) = \sum_{k \in \mathbb{Z}} a_k \theta(x-k)$$

$$\downarrow \mathcal{F}$$

$$\hat{f}(\xi) = \hat{a}(\xi) \hat{\theta}(\xi), \quad \hat{a}(\xi) := \sum_{k \in \mathbb{Z}} a_k e^{-2\pi i k \xi}$$

\hookrightarrow 1-periodic!

$$\text{Now, } \|f\|^2 = \|\hat{f}\|^2 = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi$$

$$= \int_{-1/2}^{1/2} \sum_{l \in \mathbb{Z}} |\hat{a}(\xi + l)|^2 |\hat{\theta}(\xi + l)|^2 d\xi$$

$\hat{a}(\xi)$: 1-periodic \rightarrow

$$= \int_{-1/2}^{1/2} |\hat{a}(\xi)|^2 \sum_{l \in \mathbb{Z}} |\hat{\theta}(\xi + l)|^2 d\xi$$

Hence, together with the assumption, we have

$$\frac{1}{B} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{a}(\xi)|^2 d\xi \leq \|f\|^2 \leq \frac{1}{A} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{a}(\xi)|^2 d\xi$$

But $\int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{a}(\xi)|^2 d\xi = \sum_{k \in \mathbb{Z}} |a_k|^2$ via Parseval

Hence, $A \|f\|^2 \leq \sum_{k \in \mathbb{Z}} |a_k|^2 \leq B \|f\|^2 \quad \checkmark$

(\Rightarrow) If $\{\theta(x-k)\}_{k \in \mathbb{Z}}$ is a Riesz basis for V_0 , then $\exists B \geq A > 0$ s.t.

$$A \|f\|^2 \leq \sum_{k \in \mathbb{Z}} |a_k|^2 \leq B \|f\|^2$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{a}(\xi)|^2 d\xi$$

Reversing the argument, we get

$$\frac{1}{B} \leq \sum_{k \in \mathbb{Z}} |\hat{\theta}(\xi - k)|^2 \leq \frac{1}{A}, \quad \forall \xi \in [-\frac{1}{2}, \frac{1}{2}].$$

Example 1. $\theta(x) = \chi_{[0,1)}(x)$

\Rightarrow Piecewise constant approximation

$$\theta_{j,k}(x) = 2^{-j/2} \chi_{[k2^j, (k+1)2^j)}(x)$$

$\{\theta_{j,k}\}_{k \in \mathbb{Z}}$ form an ONB for V_j .

Example 2. $\theta(x) = \text{sinc}(x) = \frac{\sin \pi x}{\pi x}$

$$\Leftrightarrow \hat{\theta}(\xi) = \chi_{[-\frac{1}{2}, \frac{1}{2})}(\xi)$$

This is also called the **Shannon scaling fcn.**

$$\theta_{j,k}(x) = 2^{-j/2} \text{sinc}(2^{-j}x - k) \Leftrightarrow \hat{\theta}_{j,k}(\xi) = 2^{j/2} e^{-2\pi i k 2^j \xi} \chi_{[-2^{-j-1}, 2^{-j-1})}(\xi)$$

$\{\theta_{j,k}\}_{k \in \mathbb{Z}}$ form an ONB for $V_j = BL_{2^{-j}}(\mathbb{R})$

Band limited fcn
with bandwidth 2^{-j}

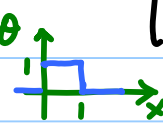
Example 3. Spline of order m (a.k.a. cardinal B-spline)

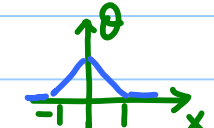
$$\theta(x) = \chi_{[0,1]} * \chi_{[0,1]} * \dots * \chi_{[0,1]}(x)$$

m times convolution

with centering at $x = \begin{cases} 0 & \text{if } m = \text{odd} \\ 1/2 & \text{if } m = \text{even} \end{cases}$

\mathcal{F}

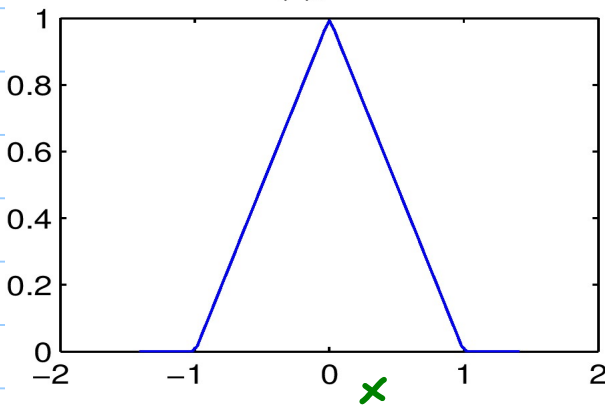
$m=0$: Box spline 

$m=1$: linear spline (hat fcn) 

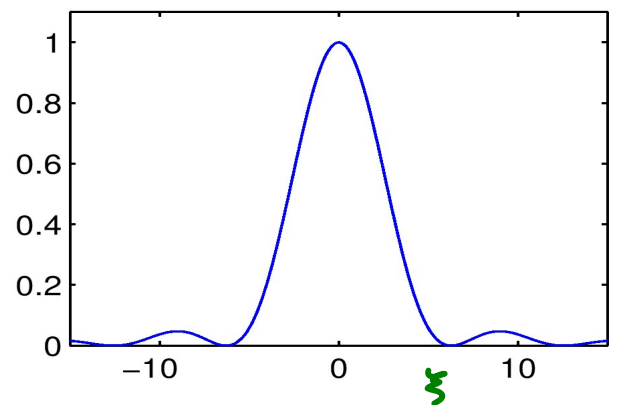
$$\hat{\theta}(\xi) = \left(\frac{\sin \pi \xi}{\pi \xi} \right)^{m+1} e^{-i \varepsilon \pi \xi} \quad \varepsilon = \begin{cases} 0 & m = \text{odd} \\ 1 & m = \text{even} \end{cases}$$

half-shift factor

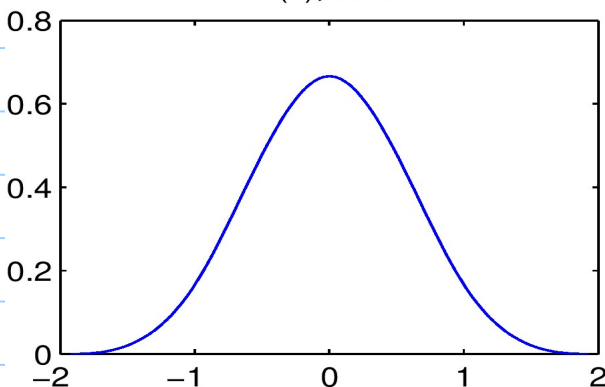
$\theta(x), m=1$



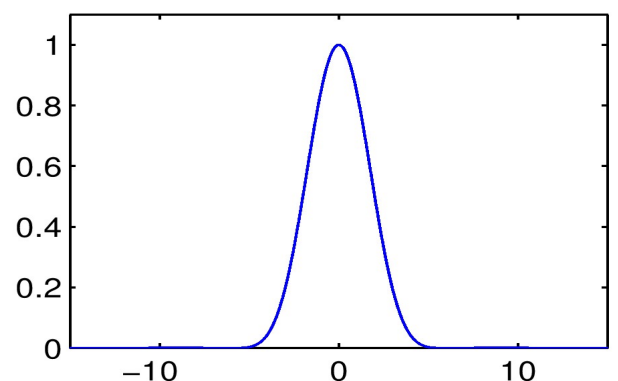
$\Theta(\xi), m=1$



$\theta(x), m=3$



$\Theta(\xi), m=3$



★ Scaling Functions (Father Wavelets)

Can we construct an ONB from the Riesz basis $\{\theta(x-k)\}_{k \in \mathbb{Z}}$ for V_0 ?

\Rightarrow Yes!

Thm (Mallat '89)

Let $\{V_j\}_{j \in \mathbb{Z}}$ be an MRA of $L^2(\mathbb{R})$.

Let $\phi(x)$ be the scaling fun whose

Fourier transf. is

$$\hat{\phi}(\xi) = \frac{\hat{\theta}(\xi)}{\left(\sum_{k \in \mathbb{Z}} |\hat{\theta}(\xi-k)|^2\right)^{1/2}}$$

division by this quantity is often called

the orthogonalization trick.

Then, $\{\phi_{j,k}\}_{k \in \mathbb{Z}}$ form an ONB for V_j , $\forall j \in \mathbb{Z}$.

(Pf) Let $\phi \in V_0$.

$$\Leftrightarrow \phi(x) = \sum_{k \in \mathbb{Z}} a_k \theta(x-k), \quad \exists \{a_k\} \in \ell^2(\mathbb{Z})$$

$$\Leftrightarrow \hat{\phi}(\xi) = \hat{a}(\xi) \hat{\theta}(\xi), \quad \hat{a}: 1\text{-periodic Fourier series and } \hat{a} \in L^2[-\frac{1}{2}, \frac{1}{2}].$$

We want to impose the orthogonality on ϕ , i.e.,

$$\langle \phi(\cdot-k), \phi(\cdot-l) \rangle = \delta_{kl}$$

$$\begin{aligned} \text{But } \langle \phi(\cdot-k), \phi(\cdot-l) \rangle &= \int_{-\infty}^{\infty} \phi(x-k) \overline{\phi(x-l)} dx \\ &= (\phi * \tilde{\phi})(l-k) \end{aligned}$$

$$\text{where } \tilde{\phi}(x) = \overline{\phi(-x)}$$

$$\text{Note: } \mathcal{F}[\phi * \tilde{\phi}](\xi) = |\hat{\phi}(\xi)|^2$$

Thus, $\{\phi(x-k)\}_{k \in \mathbb{Z}}$: an ONB

$$\iff \phi * \tilde{\phi}(k) = \delta_{k0}$$

$$\iff \sum_{k \in \mathbb{Z}} |\hat{\phi}(\xi - k)|^2 \equiv 1 \text{ a.e. } \xi \in \mathbb{R} \quad \text{--- (*)}$$

Hence if we choose

$$\hat{a}(\xi) = \left(\sum_{k \in \mathbb{Z}} |\hat{\theta}(\xi - k)|^2 \right)^{-1/2}$$

then (*) is satisfied.

$$\Rightarrow \hat{\phi}(\xi) = \frac{\hat{\theta}(\xi)}{\left(\sum_{k \in \mathbb{Z}} |\hat{\theta}(\xi - k)|^2 \right)^{1/2}}, \text{ and}$$

this is well defined since $\sqrt{A} \leq \hat{a}(\xi) \leq \sqrt{B}$ //

Using these scaling fcn's, we can easily construct P_{V_j} as

$$P_{V_j} f = \sum_{k \in \mathbb{Z}} \langle f, \phi_{j,k} \rangle \phi_{j,k}$$

$\underbrace{\qquad\qquad\qquad}_{=: f_k^j}$

$$f_k^j := \langle f, \phi_{j,k} \rangle = \int f(x) \frac{1}{\sqrt{2^j}} \phi\left(\frac{x - 2^j k}{2^j}\right) dx$$

$$= (f * \tilde{\phi}_j)(2^j k)$$

$$\tilde{\phi}_j(x) = \frac{1}{\sqrt{2^j}} \phi\left(\frac{-x}{2^j}\right) = \delta_{2^j} \tilde{\phi}(x)$$

$$\text{So, } f * \tilde{\phi}_j \xrightarrow{\mathcal{F}} \hat{f} \cdot \delta_{2^j} \hat{\phi} = \hat{f}(\xi) \cdot \sqrt{2^j} \overline{\hat{\phi}(2^j \xi)}$$

$\Rightarrow f * \tilde{\phi}_j(2^j k)$ means low pass filtering f followed by sampling at the rate 2^j .

Example 1. Haar's scaling fcn

Example 2. Shannon's scaling fcn

In these two cases, $\theta = \phi$ since $\{\theta(x-k)\}_{k \in \mathbb{Z}}$ are already orthonormal. //

Example 3. Spline scaling fcn.

$$\hat{\phi}(\xi) = \frac{e^{-i\varepsilon\pi\xi}}{\xi^{m+1} \sqrt{S_{2m+2}(\xi)}}, \quad \varepsilon = \begin{cases} 0 & m = \text{odd} \\ 1 & m = \text{even} \end{cases}$$

where $S_n(\xi) := \sum_{k \in \mathbb{Z}} (\xi+k)^{-n}$

$$\underline{m=0}: S_2(\xi) = \sum_{k \in \mathbb{Z}} (\xi+k)^{-2} = \pi^2 \operatorname{cosec}^2 \pi \xi = \left(\frac{\pi}{\sin \pi \xi} \right)^2$$

$$\Rightarrow \hat{\phi}(\xi) = e^{-i\pi\xi} \frac{1}{\xi} \cdot \frac{\sin \pi \xi}{\pi} = e^{-i\pi\xi} \operatorname{sinc}(\xi)$$

$$\underline{m=1}: S_4(\xi) = \sum_{k \in \mathbb{Z}} (\xi+k)^{-4}$$

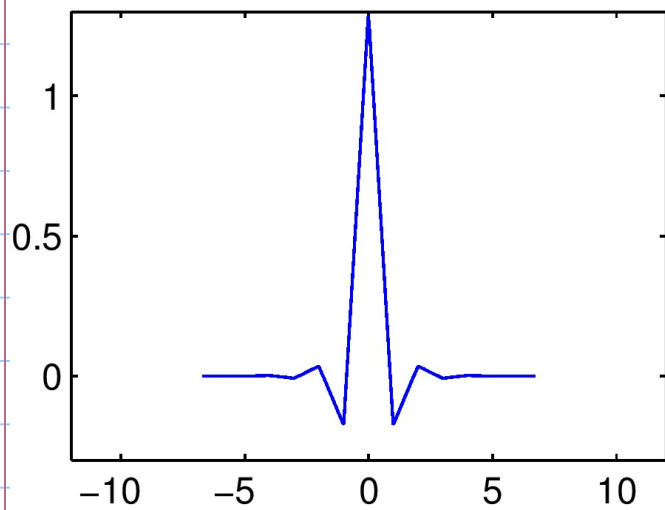
how to get this?

$$\Rightarrow S_2''(\xi) = 6 \sum_{k \in \mathbb{Z}} (\xi+k)^{-4}, \quad \text{i.e.,}$$

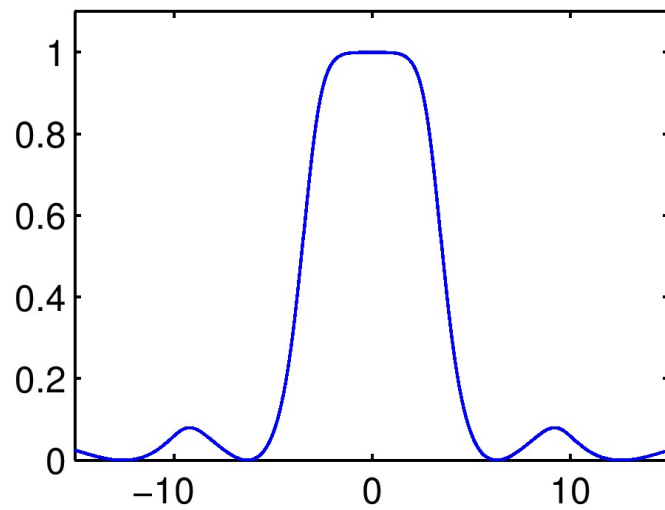
$$S_4(\xi) = \frac{1}{6} S_2''(\xi) = \dots = \frac{\pi^4}{3} \frac{1 + 2 \cos^2 \pi \xi}{\sin^4 \pi \xi}$$

$$\Rightarrow \hat{\phi}(\xi) = \frac{\sqrt{3}}{\sqrt{1 + 2 \cos^2 \pi \xi}} \left(\frac{\sin \pi \xi}{\pi \xi} \right)^2 //$$

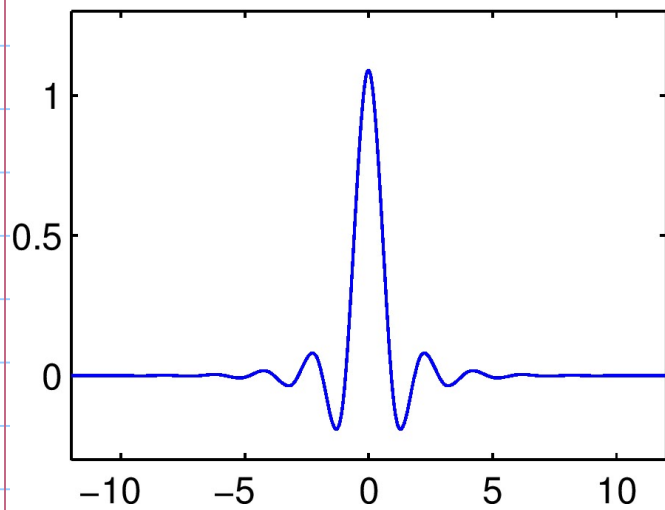
$\phi(x), m=1$



$\Phi(\xi), m=1$



$\phi(x), m=3$



$\Phi(\xi), m=3$

