

# MAT 271: Applied & Computational Harmonic Analysis Supplementary Notes III by Naoki Saito

## The Discrete Fourier Transform (DFT)

- The DFT can be viewed as either an approximation to the Fourier transform or an approximation to the Fourier series coefficients.
- Suppose  $f \in L^2[-A/2, A/2]$ , and  $f(x) = 0$  for  $|x| > A/2$ . That is,  $f$  is a space-limited, square integrable function, which is a reasonable assumption in practice. Then, we can invoke the dual version of the Shannon-Whittaker sampling theorem in the frequency domain, which can also be stated in terms of Fourier coefficients of the Fourier series (Recall that the Fourier transform of the periodic functions gives the line spectrum in the frequency domain). In fact, we have the following relationship:

$$\hat{f}(k/A) = \int_{-A/2}^{A/2} f(x) e^{-2\pi i k x / A} dx = \langle f, e^{2\pi i k \cdot / A} \rangle = \sqrt{A} \alpha_k, \quad k \in \mathbb{Z}. \quad (1)$$

- In general,  $f \in L^2[-A/2, A/2]$  is not a band-limited function. Therefore, to have a finite length vector representing the frequency samples (or the Fourier coefficients), we need to truncate the frequency information for  $|\xi| > \Omega/2$  for some  $\Omega > 0$ . This is the *first* source of error of DFT approximation to FT/FS. This truncation allows us to consider only  $k$  with  $|k| \leq A\Omega/2$ .
- We now need to approximate the Fourier integration in (1) numerically. We use the *trapezoid rule*. Here is the *second* source of the error of DFT. Let's divide the interval  $[-A/2, A/2]$  into  $N$  (*positive even integer*<sup>1</sup>) subintervals of equal length of  $\Delta x = A/N$ . Let  $x_\ell = \ell \Delta x$ ,  $\ell = (-N/2) : (N/2)$  be the points used in the trapezoid rule. Let  $g(x) = f(x) e^{-2\pi i k x / A}$ . Then we have

$$\hat{f}(k/A) \approx \frac{\Delta x}{2} \left\{ g(-A/2) + 2 \sum_{\ell=-N/2+1}^{N/2-1} g(x_\ell) + g(A/2) \right\}.$$

If we assume  $f(-A/2) = f(A/2)$  (which we should do if possible by windowing or zero-padding), then the above approximation is simplified:

$$\hat{f}(k/A) \approx \Delta x \sum_{\ell=-N/2}^{N/2-1} g(x_\ell) = \frac{A}{N} \sum_{\ell=-N/2}^{N/2-1} f(\ell A/N) e^{-2\pi i k \ell / N},$$

- Now, let  $f_\ell = f(\ell A/N)$ . Then, the  $N$ -point DFT is defined as follows:

$$F_k := \frac{1}{\sqrt{N}} \sum_{\ell=-N/2}^{N/2-1} f_\ell e^{-2\pi i k \ell / N}, \quad k = -N/2, \dots, N/2 - 1. \quad (2)$$

The factor  $1/\sqrt{N}$  is to make DFT a unitary transformation (i.e.,  $\ell^2$ -norm (energy) preserving transformation, so that the Parseval & Plancherel equalities holds.)<sup>2</sup>

<sup>1</sup>All the subsequent matrix representations assume this. See [2, Sec. 3.1] for  $N$  being positive odd integer as well as the other cases, e.g., different starting and ending indices.

<sup>2</sup>Note that the definition used in the standard book [2] uses the factor  $1/N$  instead, which makes DFT non-unitary. You need to be careful about the definition of DFT when you read literature and use software packages. More about this in the end of this note.

We now have the following relationship.

$$\hat{f}(k/A) = \sqrt{A}\alpha_k \approx \frac{A}{\sqrt{N}}F_k.$$

The  $N$ -point inverse DFT is defined, as you can imagine, as follows.

$$f_\ell := \frac{1}{\sqrt{N}} \sum_{k=-N/2}^{N/2-1} F_k e^{2\pi i k \ell / N}, \quad \ell = -N/2, \dots, N/2-1.$$

The proof of this formula gets easier when we use the vector-matrix notation later in this note.

- **[The reciprocity relations]** Let  $\Delta\xi$  be a sampling rate in the frequency domain, i.e.,  $\Delta\xi = 1/A$ . Since we know  $\Delta x = A/N$ , and  $k/A = \Omega/2$  at  $k = N/2$  (highest frequency in consideration), we have the following fundamental relations:

$$\Delta x \Delta \xi = \frac{1}{N}, \quad A\Omega = N.$$

Interpretation of these relations is very important. For example, fix  $N$ . Then increasing the length  $A$  implies increasing  $\Delta x$ , decreasing  $\Omega$ , and decreasing  $\Delta\xi$  (finer frequency sampling, but the frequency bandwidth also decreases). If we fix  $A$ , then increasing  $N$  (finer space sampling) implies decreasing  $\Delta x$  and increasing  $\Omega$  while  $\Delta\xi$  is kept constant (increasing the frequency bandwidth).

- **[A vector-matrix notation of DFT]** We can gain great insights by expressing DFT using vector-matrix notation. To do this, we need to define a couple of things. Let  $\omega_N := e^{2\pi i / N}$ , i.e.,  $N$ th root of unity. Note that  $\bar{\omega}_N = \omega_N^{-1}$ ,  $\omega_N^0 = \omega_N^N = 1$ ,  $\omega_N^{N/2} = -1$ , and  $\omega_N^{k+N} = \omega_N^k$  for any  $k \in \mathbb{Z}$ . Then, define a column vector

$$\mathbf{w}_N^k := \frac{1}{\sqrt{N}} \left( \omega_N^{k \cdot 0}, \omega_N^{k \cdot 1}, \dots, \omega_N^{k \cdot N/2}, \dots, \omega_N^{k \cdot (N-1)} \right)^T, \quad k = 0, \dots, N-1.$$

We also define another column vector

$$\tilde{\mathbf{w}}_N^k := \frac{1}{\sqrt{N}} \left( \omega_N^{k \cdot (-N/2)}, \omega_N^{k \cdot (-N/2+1)}, \dots, \omega_N^{k \cdot 0}, \dots, \omega_N^{k \cdot (N/2-1)} \right)^T, \quad k = -N/2, \dots, N/2-1.$$

Using the properties of  $\omega_N$  listed above, one can easily show that

$$\tilde{\mathbf{w}}_N^k = S_N \mathbf{w}_N^k,$$

where  $S_N$  is equivalent to `fftshift` in MATLAB, so its matrix representation is

$$S_N := \begin{bmatrix} O_{N/2} & I_{N/2} \\ I_{N/2} & O_{N/2} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \\ 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Note that the above matrix representation only holds if  $N$  is an *even* integer, which is our assumption here. In that case, we also have  $S_N^T = S_N^{-1} = S_N$ . However, if  $N$  is an odd integer, this relationship

does not hold (derive the matrix representation of  $S_N$  for odd  $N$  as an exercise!) Hence, in general, it is safer to use `ifftshift` in MATLAB to undo `fftshift`.

Let  $\mathbf{f} = (f_{-N/2}, \dots, f_{N/2-1})^T$  be a vector of sampled points  $f_\ell = f(\ell\Delta x)$ ,  $\ell = -N/2, \dots, N/2-1$ . Now DFT can be written as follows:

$$F_k = \langle \mathbf{f}, \tilde{\mathbf{w}}_N^k \rangle, \quad k = -N/2, \dots, N/2-1.$$

Finally, define an  $N$ -point DFT matrix commonly used in the literature

$$W_N := \left[ \begin{array}{c|c|c|c} \mathbf{w}_N^0 & \mathbf{w}_N^1 & \dots & \mathbf{w}_N^{N-1} \end{array} \right]$$

whereas we define the following matrix compliant with our definition of DFT (2):

$$\tilde{W}_N := \left[ \begin{array}{c|c|c|c} \tilde{\mathbf{w}}_N^{-N/2} & \tilde{\mathbf{w}}_N^{-N/2+1} & \dots & \tilde{\mathbf{w}}_N^{N/2-1} \end{array} \right] = S_N W_N S_N^T.$$

Let  $\mathbf{F} = (F_{-N/2}, \dots, F_{N/2-1})^T \in \mathbb{C}^N$ . Then, the  $N$ -point DFT/IDFT can be conveniently written as:

$$\mathbf{F} = \tilde{W}_N^* \mathbf{f}, \quad \mathbf{f} = \tilde{W}_N \mathbf{F},$$

where  $\tilde{W}_N^*$  is an hermitian conjugate (transposition followed by element-wise complex conjugation) of  $\tilde{W}_N$ , and also often written as  $\tilde{W}_N^H$  in literature. In fact,  $\tilde{W}_N^* = (S_N W_N S_N^T)^* = S_N W_N^* S_N^T$ . We also denote  $\mathcal{D}_N[\mathbf{f}] := \tilde{W}_N^* \mathbf{f}$ .

- **[Theorem 1]** Both  $W_N$  and  $\tilde{W}_N$  are  $N$ -by- $N$  unitary matrix. In other words, both  $\{\mathbf{w}_N^k\}_{k=0}^{N-1}$  and  $\{\tilde{\mathbf{w}}_N^k\}_{k=-N/2}^{N/2-1}$  are orthonormal bases of  $\mathbb{C}^N$ .  
(Proof) Exercise. A main thing is to prove  $\langle \mathbf{w}_N^k, \mathbf{w}_N^\ell \rangle = \delta_{k,\ell}$ .
- **[Theorem 2]** All the eigenvalues of  $W_N$  and  $\tilde{W}_N$  are  $1, -1, i, -i$ .  
(Proof) See [1, 2, 3]. Note that from this theorem we have  $W_N^4 = \tilde{W}_N^4 = I_N$ .
- **[Pictorial view of the matrix  $W_N^*$ ]** Using the properties of  $\omega_N$ , in particular the periodicity with

period  $N$ , we have

$$\begin{aligned}
 W_N^* &= \begin{bmatrix} (\mathbf{w}_N^0)^* \\ (\mathbf{w}_N^1)^* \\ (\mathbf{w}_N^2)^* \\ \vdots \\ (\mathbf{w}_N^{N/2})^* \\ \vdots \\ (\mathbf{w}_N^{N-1})^* \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_N^{-1} & \omega_N^{-2} & \dots & \omega_N^{-(N-1)} \\ 1 & \omega_N^{-2} & \omega_N^{-4} & \dots & \omega_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \omega_N^{-N/2} & \omega_N^{-2N/2} & \dots & \omega_N^{-(N-1)N/2} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \omega_N^{-N-1} & \omega_N^{-2(N-1)} & \dots & \omega_N^{-(N-1)(N-1)} \end{bmatrix} \\
 &= \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_N^{-1} & \omega_N^{-2} & \dots & \omega_N^{-(N-1)} \\ 1 & \omega_N^{-2} & \omega_N^{-4} & \dots & \omega_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \omega_N^{-N/2+1} & \omega_N^{2(-N/2+1)} & \dots & \omega_N^{(N-1)(-N/2+1)} \\ 1 & \omega_N^{-N/2} & \omega_N^{-2N/2} & \dots & \omega_N^{-(N-1)N/2} \\ 1 & \omega_N^{-N/2-1} & \omega_N^{2(N/2-1)} & \dots & \omega_N^{(N-1)(N/2-1)} \\ 1 & \omega_N^{-N/2-2} & \omega_N^{2(N/2-2)} & \dots & \omega_N^{(N-1)(N/2-2)} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \omega_N^{-1} & \omega_N^{-2} & \dots & \omega_N^{-(N-1)} \end{bmatrix}.
 \end{aligned}$$

The following figure shows the matrix  $W_N^*$  and  $\widetilde{W}_N^*$  with  $N = 16$  as waveforms. Note that the first row vector of a matrix is displayed in the bottom while the last row in the top in each figure. Note the change of the locations of the basis vectors as well as symmetry  $(W_N^*)^T = W_N^*$ ,  $(\widetilde{W}_N^*)^T = \widetilde{W}_N^*$ .

- **[Different forms of DFT]** It is amazing to know that the definition of DFT varies with the software systems one uses. We should always be careful what is the exact definition of the DFT for each software system.

**MATLAB, R, S-Plus:**  $F_k = \sum_{\ell=1}^N f_\ell e^{-2\pi i(k-1)(\ell-1)/N}$  for  $k = 1 : N$ .

**Mathematica:**  $F_k = \frac{1}{\sqrt{N}} \sum_{\ell=1}^N f_\ell e^{2\pi i(k-1)(\ell-1)/N}$  for  $k = 1 : N$ .

**Maple:**  $F_{k+1} = \frac{1}{N} \sum_{\ell=0}^{N-1} f_\ell e^{-2\pi i k \ell / N}$  for  $k = 0 : (N-1)$ .

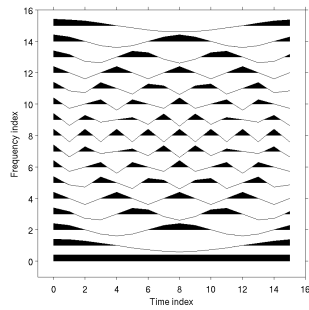
**MathCad:**  $F_k = \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} f_\ell e^{2\pi i k \ell / N}$  for  $k = 0 : (N-1)$ .

- **[Further caution]**

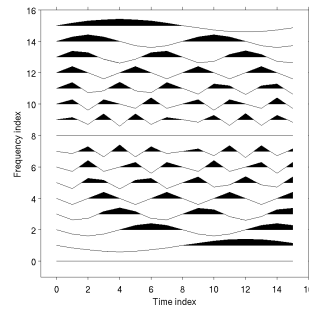
- If an input argument to the DFT/FFT function is a *matrix* (or *multidimensional array*), then MATLAB applies DFT on *each column vector for a matrix* (or *the first non-singleton dimension for a 3D or higher dimensional array*).
- On the other hand, the DFT functions in the other packages perform the *multidimensional DFT* on the input.

Hence, the DFT we defined in this note, i.e.,  $\mathbf{F} = \widetilde{W}_N^* \mathbf{f}$ , can be realized by the following MATLAB command (assuming that  $\mathbf{f}$  is a 1D vector):

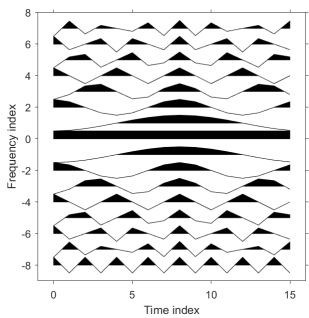
```
F=fftshift(fft(ifftshift(f)))/sqrt(length(f));
```



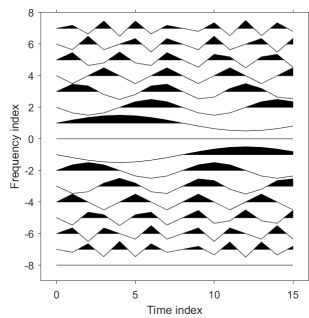
(a)  $\text{Re}(W^*)$



(b)  $\text{Im}(W^*)$



(c)  $\text{Re}(\tilde{W}^*)$



(d)  $\text{Im}(\tilde{W}^*)$

For more information about the DFT including higher-dimensional versions, see [2]. Also, the DFT matrix has more *profound* properties. See the challenging and deep paper by [1].

## References

- [1] L. AUSLANDER AND R. TOLIMIERI, *Is computing with the finite Fourier transform pure or applied mathematics?*, Bull. Amer. Math. Soc., 1 (1979), pp. 847–897.
- [2] W. L. BRIGGS AND V. E. HENSON, *The DFT: An Owner's Manual for the Discrete Fourier Transform*, SIAM, Philadelphia, PA, 1995.
- [3] J. H. MCCLELLAN AND T. W. PARKS, *Eigenvalue and eigenvector decomposition of the discrete Fourier transform*, IEEE Trans. Audio Electroacoust., AU-20 (1972), pp. 66–74. See also comments appeared in AU-21, pp. 65, 1973.