

# Lecture 2: Basics of Fourier Transforms

Note Title

1/7/2014

We will mainly focus on 1D signals.

## 1. Fourier Transform on $L^1(\mathbb{R})$

Let  $f \in L^1(\mathbb{R}) := \{f: \mathbb{R} \rightarrow \mathbb{C} \mid \|f\|_1 < \infty\}$

where  $\|f\|_1 := \int_{-\infty}^{\infty} |f(x)| dx$  is called

the  $L^1$ -norm of  $f$ . This integral is that of Lebesgue

Def. The **Fourier transform** of  $f \in L^1(\mathbb{R})$  is the function  $\hat{f}$  on  $\mathbb{R}$  defined by

$$(*) \quad \hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx$$

We also write  $\mathcal{F}[f](\xi) = \hat{f}(\xi)$

(Note that in  $\mathbb{R}^d$ , the definition becomes)

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle \xi, x \rangle} dx$$

Now, (\*) is well defined, i.e., abs. conv.:

$$|\hat{f}(\xi)| \leq \int_{-\infty}^{\infty} |f(x)| dx = \|f\|_1 < \infty$$

since  $|e^{-2\pi i \xi x}| = 1$ . //

In fact,  $\mathcal{F}: L^1(\mathbb{R}) \rightarrow BC(\mathbb{R}) = C(\mathbb{R}) \cap L^\infty(\mathbb{R})$

## \* Basic Properties of the Fourier transform

Def. For any  $a \in \mathbb{R}$ , the **translation operator**  $T_a$  is defined by

$$T_a f(x) := f(x - a)$$

Def. For any  $s > 0$ , the **dilation operator**  $\mathcal{D}_s$  is defined by

$$\mathcal{D}_s f(x) = \frac{1}{\sqrt{s}} f\left(\frac{x}{s}\right)$$

$\searrow$  This factor guarantees that  $\mathcal{D}_s$  is an **isometry** in  $L^2(\mathbb{R})$   
i.e.,  $\|f\|_2 = \|\mathcal{D}_s f\|_2$

We'll do more of  $L^2$  shortly.

$$\|f\|_2 := \sqrt{\int_{-\infty}^{\infty} |f(x)|^2 dx}$$

Def. The **convolution** of  $f, g \in L^1$ , is defined as

$$f * g(x) := \int f(x - y) g(y) dy$$

Note:  $f * g = g * f$

$(f * g) * h = f * (g * h)$ , etc.

Thm 1. Let  $f, g \in L^1$ . Then,

$$(a) \mathcal{F}[\tau_a f] = e^{-2\pi i \xi a} \hat{f}(\xi)$$

$$\mathcal{F}[e^{2\pi i a x} f(x)] = \tau_a \hat{f}(\xi) = \hat{f}(\xi - a)$$

$$(b) \mathcal{F}[\delta_s f] = \sqrt{s} \hat{f}(s\xi) = \delta_{1/s} \hat{f}(\xi)$$

clearly,  $\mathcal{F}[\delta_{1/s} f] = \delta_s \hat{f}(\xi)$

$$(c) \mathcal{F}[f * g] = \hat{f} \hat{g}$$

$$\mathcal{F}[f g] = \hat{f} * \hat{g}$$

$$(d) \text{ Let } \partial_x := \frac{\partial}{\partial x} \dots$$

If  $f \in C^k$  and  $\partial_x^j f \in L^1$ ,  $j=1, \dots, k$   
and  $\partial_x^j f \in C_0$ ,  $j=1, \dots, k-1$ ,

then

$$\mathcal{F}[\partial_x^k f] = (2\pi i \xi)^k \hat{f}(\xi)$$

$\hookrightarrow$  continuous & vanishes at  $\pm\infty$   
 $\Rightarrow$  used in integration by parts!

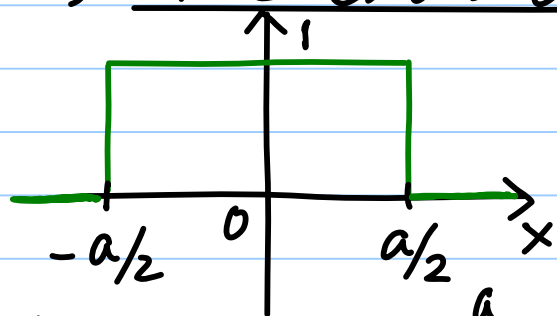
(e) On the other hand,

if  $x^j f \in L^1$ ,  $j=1, \dots, k$ ,

then  $\hat{f} \in C^k$  and  $\mathcal{F}[x^k f(x)] = \left(\frac{i}{2\pi}\right)^k \partial_\xi^k \hat{f}(\xi)$

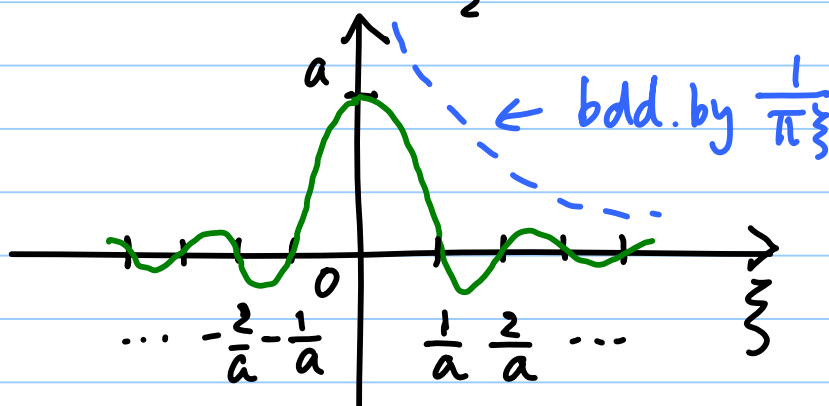
## \* Important Examples

(1) The characteristic (indicator) fun



$$\chi_{(-\frac{a}{2}, \frac{a}{2})}(x) := \begin{cases} 1 & \text{if } |x| < \frac{a}{2} \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{aligned} \hat{\chi}_{(-\frac{a}{2}, \frac{a}{2})}(\xi) &= \int_{-\frac{a}{2}}^{\frac{a}{2}} 1 \cdot e^{-2\pi i \xi x} dx = \frac{e^{-\pi i a \xi} - e^{\pi i a \xi}}{-2\pi i \xi} \\ &= \frac{\sin \pi a \xi}{\pi \xi} \end{aligned}$$



zeros are at  
 $\xi = n/a, n \in \mathbb{Z} - \{0\}$ .

If we define  $\text{sinc}(x) := \frac{\sin \pi x}{\pi x}$ ,

then we can write

$$\hat{\chi}_{(-\frac{a}{2}, \frac{a}{2})}(\xi) = a \text{sinc}(a\xi)$$

Note that this fun is not in  $L^1$  but in  $L^2$ !

$$\odot \int_{-\infty}^{\infty} \left| \frac{\sin \pi a \xi}{\pi \xi} \right| d\xi = +\infty \text{ but } \int_{-\infty}^{\infty} \left| \frac{\sin \pi a \xi}{\pi \xi} \right|^2 d\xi = a < \infty$$

The details are left as an exercise. //

(2) The Gaussian fcn

$$g(x; \sigma) = g_\sigma(x) := \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}, \quad \sigma > 0$$

$x \in \mathbb{R}$

How to compute  $\hat{g}_\sigma(\xi)$ ?

$\Rightarrow$  HW problem!  $\exists$  a neat way.

$$\hat{g}_\sigma(\xi) = e^{-2\pi^2\sigma^2\xi^2}$$

Incidentally, in general, we have

$$\hat{f}(0) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \cdot 0 \cdot x} dx = \int_{-\infty}^{\infty} f(x) dx$$

= The DC component of  $f(x)$   
 $\swarrow$  EE terminology

Hence,  $\hat{g}_\sigma(0) = 1$ , i.e.,  $g_\sigma(x)$  is a probability density fcn (pdf)

Also note that if  $\sigma = \frac{1}{\sqrt{2\pi}}$ , then

$$e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi \xi^2}$$

invariant w.r.t.  $\mathcal{F}$ !

## \* The Riemann-Lebesgue Lemma

If  $f \in L^1$ , then  $\hat{f}(\xi) \rightarrow 0$  as  $\xi \rightarrow \pm\infty$

(i.e.,  $\mathcal{F}[L^1(\mathbb{R})] \subset C_0(\mathbb{R})$ )

"If  $f \in L^1$ , very high frequency components vanish." (non rigorous statement!)

## \* The Fourier Inversion Thm

states a procedure to recover  $f$  from  $\hat{f}$ .

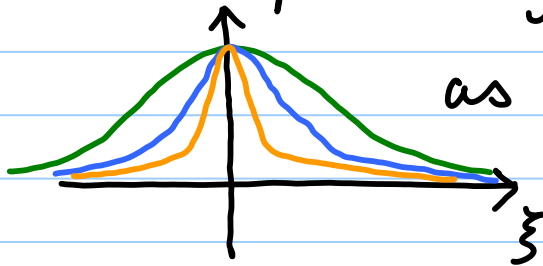
Want to have

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi$$

But the problem is:

$\hat{f}$  may not be in  $L^1$ . e.g.,  $\text{sinc} \notin L^1$

Hence, we use the "cut off" fcn  
a good example is  $g_a(\xi) = e^{-2\pi^2 a^2 \xi^2}$



as  $a \uparrow$ , more localized

Consider

$$(*) \int \underbrace{\hat{f}(\xi)} \hat{g}_\alpha(\xi) e^{2\pi i \xi x} d\xi$$
$$= \iint \underbrace{f(y) e^{-2\pi i \xi y}} dy e^{-2\pi^2 \sigma^2 \xi^2} e^{2\pi i \xi x} d\xi$$

OK  
by  
Fubini

$$= \iint f(y) e^{2\pi i \xi (x-y)} e^{-2\pi^2 \sigma^2 \xi^2} d\xi dy$$

$$\text{Now } \int e^{2\pi i \xi (x-y)} e^{-2\pi^2 \sigma^2 \xi^2} d\xi$$
$$= \mathcal{F}_\xi^{-1} [e^{-2\pi^2 \sigma^2 \xi^2}] (y-x)$$
$$= \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(y-x)^2}{2\sigma^2}} = g_\sigma(y-x)$$
$$= g_\sigma(x-y)$$

$$\text{So } (*) = \int f(y) g_\sigma(x-y) dy = f * g_\sigma(x)$$

We can show that

$$\|f * g_\sigma - f\|_1 \rightarrow 0 \text{ as } \sigma \rightarrow 0$$

$$\text{and } \lim_{\sigma \rightarrow 0} f * g_\sigma = f \text{ a.e.}$$

almost everywhere

So, if  $\hat{f} \in L^1$ , then things become easy!

Def. Define  $\check{f}(x) := \mathcal{F}^{-1}[f(\xi)](x)$

$$= \int_{-\infty}^{\infty} f(\xi) e^{2\pi i \xi x} d\xi$$

This is the inverse Fourier transf.  
of  $f \in L^1$

Thm 2. If  $f$  &  $\hat{f}$  are both in  $L^1$ ,  
then  $(\hat{f})^\vee = (\check{f})^\wedge = f$  a.e.

Cor. If  $\hat{f} = \hat{g}$ , then  $f = g$  a.e.  
 $\hat{f} = 0$ ,  $f = 0$  a.e.

• Many fcn in  $L^1$  has their F.T. in  $L^1$ ,  
but not all of the  $L^1$  fcn have  $\hat{f} \in L^1$ .

For  $f \in L^1$  to have  $\hat{f} \in L^1$ ,  $f$  must be  
a little smooth.

$\Rightarrow \hat{f}(\xi)$  must decay as  $|\xi| \uparrow$ .

e.g., if  $|\hat{f}(\xi)| \leq \frac{C}{1+|\xi|^2}$  for  $\exists C > 0$ .  
then  $\hat{f} \in L^1$  (easy to show.)

If  $f \in C^2$  &  $f', f'' \in L^1$ , then  $f$  decays as  
above.

" Smoothness of a fcn

$\Leftrightarrow$  Decay of the Fourier  
transf. at high freq. "



In particular, if  $\hat{f}$  has a compact support, then it's great!

Such  $f$  is called a **band-limited** fcn.

## 2. The Fourier Transform on $L^2$

So far, we have dealt with FT on  $L^1$ .

Simply assuming  $f \in L^2(\mathbb{R})$ ,

$\int f(x) e^{-2\pi i \xi x} dx$  may not converge.

e.g.,  $f(x) = \text{sinc}(x) \in L^2$   
but it is **not** in  $L^1$ .

• We will overcome this problem as follows.

Define a subspace  $\mathcal{X} \subset L^1$  s.t.

$$\mathcal{X} := \{f \in L^1 \mid \hat{f} \in L^1\}$$

For any  $f \in \mathcal{X}$ , both  $f$  &  $\hat{f}$  are in  $BC(\mathbb{R})$   
so,  $\mathcal{X} \subset L^2$  (because  $f \in L^1 \cap BC$ )

$\Rightarrow f \in L^2$ , which comes from the general

Thm:  $L^p \cap L^r \subset L^q$  for

set  $p=1, q=2, r=\infty$ .  $\left( 0 < p < q < r \leq \infty. \right)$   
(can be proved via Hölder's ineq.)

This general thm also implies

$$\|f\|_q \leq \|f\|_p \vee \|f\|_r \\ = \max(\|f\|_p, \|f\|_r)$$

Also, you can show that  $\mathcal{X}$  is **dense** in  $L^2$ .

- Hence, for any  $f \in L^2$ , we can find a sequence  $\{f_n\} \subset \mathcal{X}$  s.t.  $\|f_n - f\|_2 \xrightarrow{n \rightarrow \infty} 0$
- $\{f_n\} \subset \mathcal{X}$  implies  $\{\hat{f}_n\} \subset \mathcal{X}$  the Plancherel eq.

Now, we can show  $\|\hat{f}_n - \hat{f}_m\|_2 \stackrel{\downarrow}{=} \|f_n - f_m\|_2 \rightarrow 0$  as  $n, m \rightarrow \infty$

That is,  $\{f_n\}$  is a **Cauchy sequence** in  $L^2$ .

Since  $L^2$  is **complete**,  $\hat{f}_n$  has a limit  $\hat{f}$  in  $L^2$ , which we define this limit as  $\hat{f}$ , the Fourier transf. of  $f$ .

Parseval is for Fourier series. \* The Plancherel Thm ( $\mathcal{F}$  is an isometry in  $L^2$ )

For any  $f, g \in L^2$ ,

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle, \text{ hence } \|f\|_2 = \|\hat{f}\|_2$$

- Finally, we can do the following. Suppose  $\phi(x) \in L^2$ . Then  $\exists \hat{f} \in L^2$  s.t.  $\phi(x) = \hat{f}(x)$ . Then,  $\phi(\xi) = f(-\xi)$ . Ex:  $\phi(x) = \text{sinc}(x) \Rightarrow \hat{\phi}(\xi) = \chi_{(-\frac{1}{2}, \frac{1}{2})}(\xi)$ .