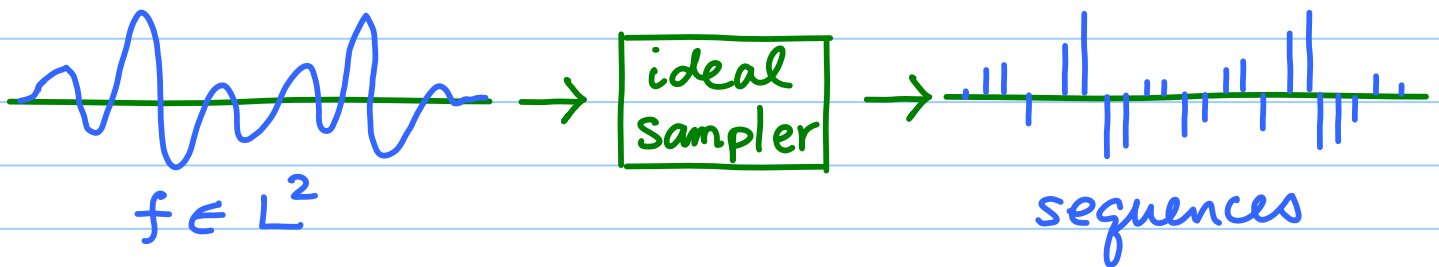


# Lecture 4: Discretization via Sampling

Note Title

1/20/2014

- The Riemann-Lebesgue Lemma says the high freq. components attenuates in  $L^1$ .
  - We know from our daily experience that the high freq. info is difficult to transmit / propagate.
- ⇒ Band-limited fns are common.



Bottom line question:

when we obtain sample values of  $f(x)$  at  $x_1 < x_2 < \dots$ , how much information of  $f$  we retain or lose?

In general,  $\{f(x_k)\}_{k \in \mathbb{N}}$  do **not** tell anything about  $f(x)$  for  $x \notin \{x_k\}_{k \in \mathbb{N}}$ .

However, if  $f \in$  a space of band-limited fns  $\subset L^2$ , and  $\{x_k\}_{k \in \mathbb{N}}$  satisfy a certain condition, then this sequence  $\{f(x_k)\}_{k \in \mathbb{N}}$  tells you **EVERYTHING** about  $f$ !! Moreover,  $\{f(x_k)\}_{k \in \mathbb{N}}$  can be

expansion coeff's w.r.t. a specific ONB of the space of BL fns

Def.  $BL_\Omega(\mathbb{R}) := \{f \in L^2(\mathbb{R}) \mid \hat{f}(\xi) = 0 \ \forall |\xi| \geq \frac{\Omega}{2}\}$ .

## Thm (The Sampling Thm)

Suppose  $f \in BL_{\Omega}(\mathbb{R}) \subset L^2(\mathbb{R})$ .

Then  $f$  is **completely determined** by its samples at  $x = k/\Omega$ ,  $k \in \mathbb{Z}$ .

In fact, 
$$f(x) = \sum_{k \in \mathbb{Z}} f(k/\Omega) \frac{\sin \pi \Omega (x - k/\Omega)}{\pi \Omega (x - k/\Omega)}$$

$$= \sum_{k \in \mathbb{Z}} f(k/\Omega) \operatorname{sinc}(\Omega x - k)$$

Remark: This is a folklore thm. Often attributed to E.T. Whittaker (1915), British → interpolation rather than sampling  
V. Kotelnikov (1933), Russian  
H. Raabe (1939), German  
C. Shannon (1948), American  
I. Soseki (1949), Japanese

But it turned out that K. Ogura (1920) seems to be the **first** clear statement of the classical sampling thm. according to P.L. Butzer et al. (2011).

(Pf) Since  $\hat{f}(\xi)$  is supported only on  $[-\frac{\Omega}{2}, \frac{\Omega}{2}]$ , we can expand it into **the Fourier series**:

$$\hat{f}(\xi) = \sum_{k \in \mathbb{Z}} \alpha_k \frac{1}{\sqrt{\Omega}} e^{2\pi i k \xi / \Omega}, \quad |\xi| \leq \frac{\Omega}{2}.$$

$$\alpha_k = \left\langle \hat{f}, \frac{1}{\sqrt{\Omega}} e^{2\pi i k \xi / \Omega} \right\rangle$$

the  $k$ th Fourier coeff.

$$= \frac{1}{\sqrt{\Omega}} \int_{-\Omega/2}^{\Omega/2} \hat{f}(\xi) e^{-2\pi i k \xi / \Omega} d\xi$$

$$f \in BL_{\Omega}(\mathbb{R})$$

Fourier inversion thm!

$$\begin{aligned} & \downarrow \\ &= \frac{1}{\sqrt{\Omega}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-2\pi i k \xi / \Omega} d\xi \\ &= \frac{1}{\sqrt{\Omega}} f\left(-\frac{k}{\Omega}\right) \quad \text{--- (*)} \end{aligned}$$

$$\begin{aligned} \Rightarrow f(x) &= \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi \\ &= \int_{-\Omega/2}^{\Omega/2} \sum_{k \in \mathbb{Z}} \alpha_k \frac{1}{\sqrt{\Omega}} e^{2\pi i \xi k / \Omega} e^{2\pi i \xi x} d\xi \end{aligned}$$

via (\*)  
+ exchange of  $\int$  &  $\sum$

$$\begin{aligned} & \downarrow \\ &= \sum_{k \in \mathbb{Z}} \left( \frac{1}{\Omega} f\left(-\frac{k}{\Omega}\right) \right) \int_{-\Omega/2}^{\Omega/2} e^{2\pi i \xi (x + k/\Omega)} d\xi \\ &= \sum_{k \in \mathbb{Z}} \frac{1}{\Omega} f\left(-\frac{k}{\Omega}\right) \left[ \frac{e^{2\pi i \xi (x + k/\Omega)}}{2\pi i (x + k/\Omega)} \right]_{-\Omega/2}^{\Omega/2} \end{aligned}$$

$$= \sum_{k \in \mathbb{Z}} f\left(-\frac{k}{\Omega}\right) \frac{\sin \pi (\Omega x + k)}{\pi (\Omega x + k)}$$

$$\begin{aligned} & \uparrow \\ &= \sum_{k \in \mathbb{Z}} f\left(\frac{k}{\Omega}\right) \frac{\sin \pi \Omega (x - k/\Omega)}{\pi \Omega (x - k/\Omega)} \quad (**) \end{aligned}$$

exact!      expansion coef      ONB of  $BL_{\Omega}(\mathbb{R})$

So if we sample points on  $x_k = k/\Omega$ ,  $k \in \mathbb{Z}$ , then we completely know  $f(x)$  not only  $x \in \{x_k\}_{k \in \mathbb{Z}}$ , but  $\forall x \in \mathbb{R}$ !

To obtain  $f(x)$  for  $x \notin \{x_k\}_{k \in \mathbb{Z}}$ , we use (\*\*) above, which is called the **band-limited interpolation formula**.

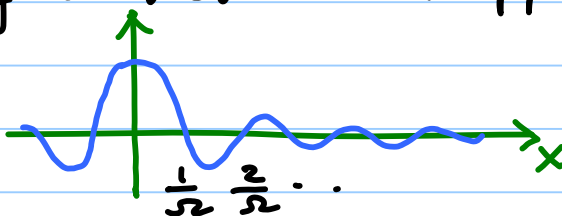
Exercise: (1) Compute  $\mathcal{F}[\text{sinc}(\Omega x - k)]$ .  
 (2) Show that  $\{\sqrt{\Omega} \text{sinc}(\Omega x - k)\}_{k \in \mathbb{Z}}$  form an ONB of  $BL_{\Omega}(\mathbb{R})$

Remarks:

• Let  $\Delta x_k := x_{k+1} - x_k = \frac{1}{\Omega} =: \Delta x$ .  
 the sampling rate  
 or the sampling interval

$\Omega \uparrow \Leftrightarrow \Delta x \downarrow$  : quite intuitive.

• A problem of sinc fcn  
 = decay is slow & support is not finite.



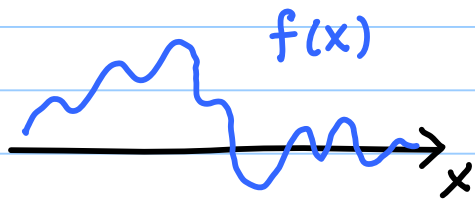
•  $\frac{1}{\sqrt{\Omega}} f\left(\frac{k}{\Omega}\right) = \langle f, \sqrt{\Omega} \text{sinc}(\Omega \cdot - k) \rangle$

•  $\sqrt{\Omega} \text{sinc}(\Omega x - k) = \delta_{\Omega^{-1}} \tau_k \text{sinc}(x)$ .  
 dilation translation

In order to interpret the essence of the Sampling Thm, we need the notion of the **generalized fcn.**

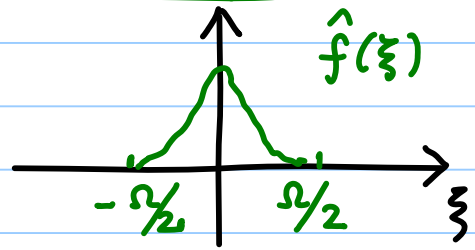
# Graphical Interpretation of the Sampling Thm.

space/time



$\mathcal{F}_c$

frequency



Dirac

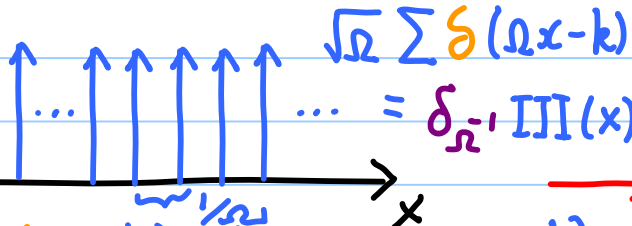
$\delta$ : delta fcn...

$\delta_{\Omega^{-1}}$ : dilation op.

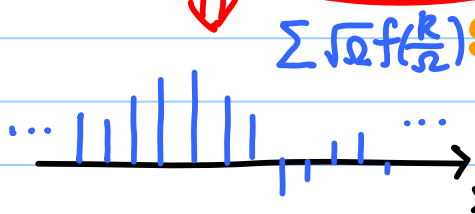
$\text{III}(x) := \sum \delta(x-k)$ , the Shah (or comb)

$\hat{\text{III}}(\xi) = \text{III}(\xi)$

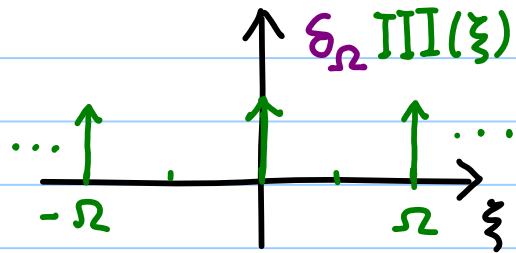
• multiplication



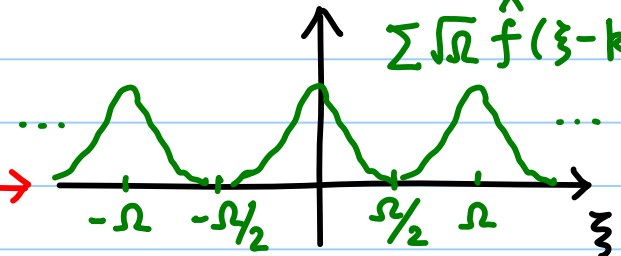
sampling



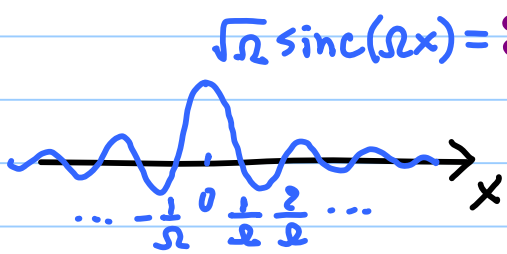
\* convolution



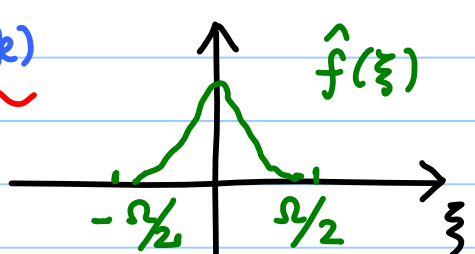
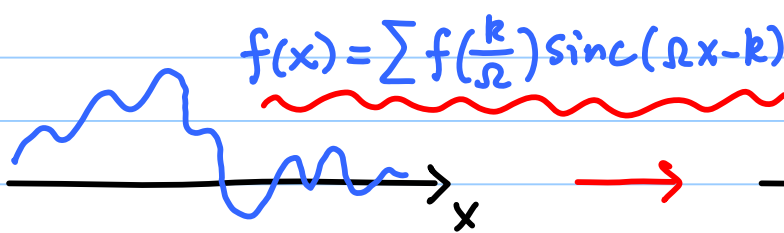
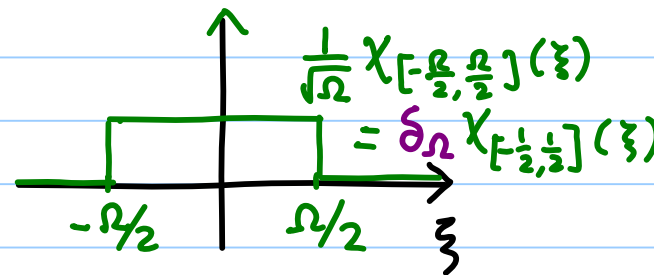
periodization



\* convolution



• multiplication



# MAT 271: Applied & Computational Harmonic Analysis Supplementary Notes I by Naoki Saito

## The Generalized Functions

- The generalized functions have more singular behavior than functions (thus the name “generalized functions”), and are always defined as *linear functionals* on the *dual space*. Thus, before we discuss the generalized functions, we need to know the following.

**Definition:** Let  $\mathcal{X}$  be a vector space over, say,  $\mathbb{C}$ . A linear map from  $\mathcal{X}$  to  $\mathbb{C}$  is called a *linear functional* on  $\mathcal{X}$ . If  $\mathcal{X}$  is a normed vector space, then the space  $\mathcal{L}(\mathcal{X}, \mathbb{C})$  of *bounded linear functionals* on  $\mathcal{X}$  is called the *dual space*, and denoted by  $\mathcal{X}^*$  (or  $\mathcal{X}'$ ).

Examples: The dual of  $L^p(\mathbb{R})$ ,  $1 < p < \infty$ , is  $L^q(\mathbb{R})$ , where  $(1/p) + (1/q) = 1$ . These numbers are called *conjugate exponents*. In particular,  $L^2$  is self dual. Similarly, the dual of the sequence space  $\ell^p(\mathbb{Z})$  is  $\ell^q(\mathbb{Z})$ .

**Hölder’s Inequality:** Let  $p$  and  $q$  are conjugate exponents. Then for any  $f \in L^p$ ,  $g \in L^q$ , we have

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

(As you can see, the Cauchy-Schwarz inequality is a special version of this with  $p = q = 2$ . The proof is a great exercise.)

**The Riesz Representation Theorem:** Suppose  $p$  and  $q$  are conjugate exponents with  $1 < p < \infty$ . Then for each linear functional  $\varphi \in (L^p)^*$ , there exists  $g \in L^q$  such that  $\varphi(f) = \int f(x)g(x) dx$  for all  $f \in L^p$ . In other words,  $(L^p)^*$  is isometrically isomorphic to  $L^q$ .

- The more singular the class of the generalized functions, the more regular its dual.
- We now define the *Schwartz class*  $\mathcal{S} := \{f \in C^\infty(\mathbb{R}) \mid \sup_{x \in \mathbb{R}} |x^k \partial^\ell f| < \infty, \text{ for any } k, \ell \in \mathbb{N}\}$ , which are very smooth and decay faster than any polynomial at infinity, i.e., a very nice class of functions. An example: The Gaussian  $g(x) = e^{-x^2}$ .
- Then, we consider the dual  $\mathcal{S}'$ . You can imagine that members of this class can be very singular or “spiky.” This dual space is called the *tempered distributions*. Being as a linear functional, each member of  $\mathcal{S}'$  acts on the Schwartz functions. More precisely, if  $F \in \mathcal{S}'$  and  $\phi \in \mathcal{S}$ , then the value of  $F$  at  $\phi$  ( $F$  is a *linear map* from  $\mathcal{S}$  to  $\mathbb{C}$ !!) is denoted as  $\langle F, \phi \rangle = F(\phi) = \int F(x)\phi(x) dx$ .
- An example: *the Dirac delta function*  $\delta(x) \in \mathcal{S}'$  is defined as  $\langle \delta, \phi \rangle = \phi(0)$ . In other words,  $\int_{-\infty}^{\infty} \delta(x)\phi(x) dx = \phi(0)$ .
- For any  $F \in \mathcal{S}'$  and any  $\phi \in \mathcal{S}$ , we can define the following operations:

**Differentiation:**  $\langle \partial_x^k F, \phi \rangle = (-1)^k \langle F, \partial_x^k \phi \rangle$ .

This can be shown by integration by parts. An example:  $\langle \delta', \phi \rangle = -\langle \delta, \phi' \rangle = -\phi'(0)$ .

**Convolution:**  $F * \phi(x) = \langle F, \tau_x \tilde{\phi} \rangle$ , where  $\tilde{\phi}(y) = \phi(-y)$ .

An example:  $(\delta * \phi)(x) = \phi(x)$ .

**Fourier transform:**  $\langle \hat{F}, \phi \rangle = \langle F, \hat{\phi} \rangle$ .

An example:  $F = \delta$ , then  $\langle \hat{\delta}, \phi \rangle = \langle \delta, \hat{\phi} \rangle = \hat{\phi}(0)$ . This essentially shows that  $\hat{\delta}(\xi) \equiv 1$ . Using the translation operator, we can also have  $\mathcal{F}\{\delta(x-a)\} = e^{-2\pi i \xi a}$ , and  $\mathcal{F}\{e^{-2\pi i x a}\} = \delta(\xi+a)$ .

- **Definition:** A tempered distribution  $F$  on  $\mathbb{R}$  is called *periodic* with period  $A$  if  $\langle F, \tau_A \phi \rangle = \langle F, \phi \rangle$  for all  $\phi \in \mathcal{S}$ . A sequence of tempered distributions  $\{F_n\}$  is said to *converge temperately* to a tempered distribution  $F$  if  $\langle F_n, \phi \rangle \rightarrow \langle F, \phi \rangle$  as  $n \rightarrow \infty$  for all  $\phi \in \mathcal{S}$ . (See that all these operations and definitions are now moved to the *nice spouses* of  $F$ !)

- **[Theorem]** If  $F$  is a periodic tempered distribution, then  $F$  can be expanded in a temperately convergent Fourier series,  $F(x) = \frac{1}{\sqrt{A}} \sum_{-\infty}^{\infty} \alpha_k e^{2\pi i k x / A}$ , i.e.,  $\langle F, \phi \rangle = \sum_{-\infty}^{\infty} \alpha_k \left\langle \frac{1}{\sqrt{A}} e^{2\pi i k \cdot / A}, \phi \right\rangle$  for all  $\phi \in \mathcal{S}$ . Moreover, the coefficients  $\alpha_k$  satisfy  $\alpha_k \leq C(1 + |k|)^N$  for some  $C, N \geq 0$ . Conversely, if  $\{\alpha_k\}$  is any sequence satisfying this estimate, the series  $\frac{1}{\sqrt{A}} \sum_{-\infty}^{\infty} \alpha_k e^{2\pi i k x / A}$  converges temperately to a periodic tempered distribution.

- Define the *Shah function* (or *comb function*),  $\text{III}_A(x) := \sum_{k=-\infty}^{\infty} \delta(x - kA)$ . The facts about this function:

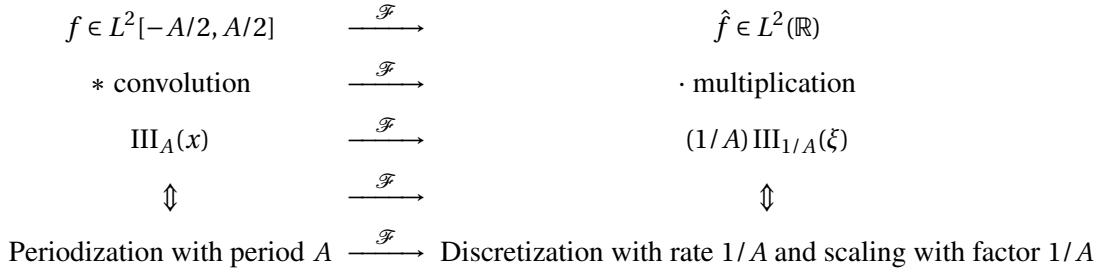
1. Since this is a periodic tempered distribution, we can expand it into the temperately convergent Fourier series;  $\text{III}_A(x) \sim \frac{1}{A} \sum_{-\infty}^{\infty} e^{2\pi i k x / A}$ . Note that  $\alpha_k \equiv 1/\sqrt{A}$  for all  $k \in \mathbb{Z}$ .

2.  $\mathcal{F}\{\text{III}_A\}(\xi) = \frac{1}{A} \text{III}_{1/A}(\xi) = \frac{1}{A} \sum_{-\infty}^{\infty} \delta(\xi - \frac{k}{A})$ .

- Using the Shah function and its Fourier transform, we can see that the Fourier transform of the Fourier series of a periodic function on  $[-A/2, A/2]$  as follows:

$$f(x) \sim \frac{1}{\sqrt{A}} \sum_{-\infty}^{\infty} \alpha_k e^{2\pi i k x / A} \xrightarrow{\mathcal{F}} \frac{1}{\sqrt{A}} \sum_{-\infty}^{\infty} \alpha_k \delta(\xi - \frac{k}{A}) \quad \text{i.e., line spectrum (discrete)}$$

As you can see, as  $A$  gets large, we are doing the finer sampling in the frequency domain, i.e.,



- **Periodization of a function with compact support  $\iff$  Discretization in frequency domain (with amplitude rescaling)**

For the details of the facts in these notes, see [1, Chap. 9], [2, Chap. 9], [3, Chap. 1].

## References

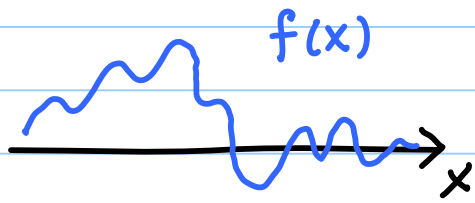
[1] G. B. FOLLAND, *Fourier Analysis and Its Applications*, Amer. Math. Soc., Providence, RI, 1992. Republished by AMS, 2009.

[2] ———, *Real Analysis: Modern Techniques and Their Applications*, John Wiley & Sons, Inc., 2nd ed., 1999.

[3] E. M. STEIN AND G. WEISS, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, 1971.

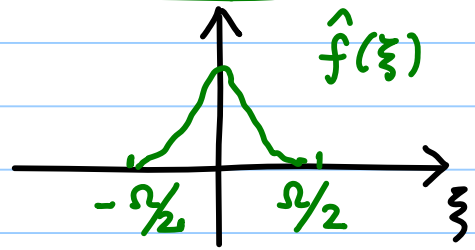
# Reprise!

space / time



$\mathcal{F}$

frequency



Dirac

$\delta$ : delta fcn...

$\delta_{\Omega^{-1}}$ : dilation op.

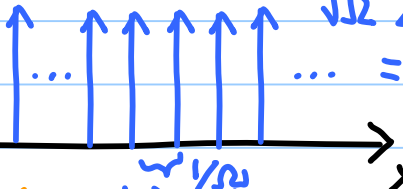
$\text{III}(x) := \sum \delta(x-k)$ , the shah (or comb)

$\hat{\text{III}}(\xi) = \text{III}(\xi)$

• multiplication

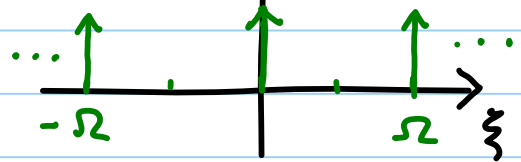
$$\sqrt{\Omega} \sum \delta(\Omega x - k)$$

$$= \delta_{\Omega^{-1}} \text{III}(x)$$



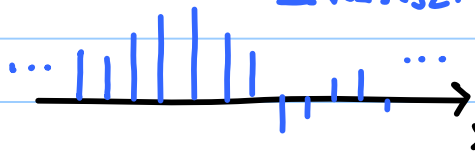
\* convolution

$$\delta_{\Omega} \text{III}(\xi)$$



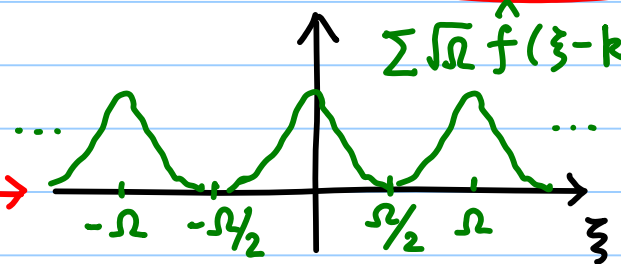
sampling

$$\sum \sqrt{\Omega} f(\frac{k}{\Omega}) \delta(\Omega x - k)$$



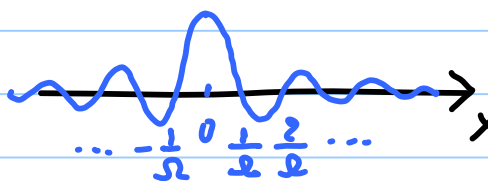
periodization

$$\sum \sqrt{\Omega} \hat{f}(\xi - k\Omega)$$



\* convolution

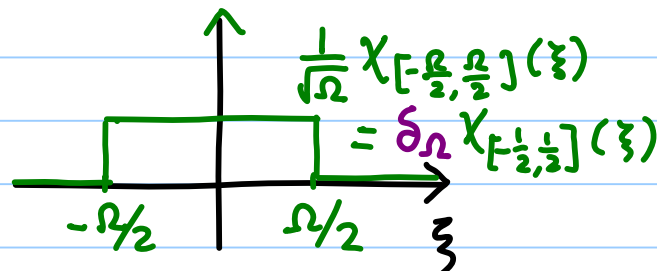
$$\sqrt{\Omega} \text{sinc}(\Omega x) = \delta_{\Omega^{-1}} \text{sinc}(x)$$



• multiplication

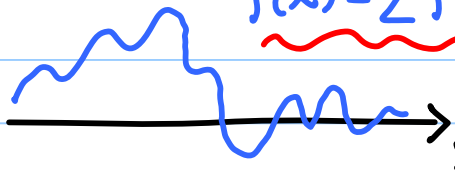
$$\frac{1}{\sqrt{\Omega}} \chi_{[-\frac{\Omega}{2}, \frac{\Omega}{2}]}(\xi)$$

$$= \delta_{\Omega} \chi_{[-\frac{1}{2}, \frac{1}{2}]}(\xi)$$



$\Downarrow$

$$f(x) = \sum f(\frac{k}{\Omega}) \text{sinc}(\Omega x - k)$$



$\Downarrow$

$$\hat{f}(\xi)$$

