☆ Functions of Bounded Variations

Why are we interested in BV ?
- Often chosen as a model for piecewise smooth signals & images
- Useful in data compression & statistical estimation
- Provide sharp info on the decay rate of the Fourier coeff's.

Let $g(x)$ be a fcn on a closed interval $I = [a, b]$. (I could be $\mathbb{R}$).
Let $D :=$ a subdivision of I, i.e.,
$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$
Now, let's form the sum :
$$T_D(g) := \sum_{k=1}^{n} |g(x_k) - g(x_{k-1})|$$

<u>Def.</u> If $T_D(g) < \infty$ for all possible subdivision D, <u>then</u> g is said to be of bdd. var. in I, and the total variation of g in I is defined as
$$V_I[g] = V_a^b[g] := \sup_D T_D(g).$$
$BV(I) :=$ a set of all fcns of bdd. var. in I.

<u>Fact</u>: $\cdot \; |g(b) - g(a)| \leq V_a^b[g] < \infty$. ← Take $x_0 = a$, $x_1 = b$.

$\cdot$ <u>If</u> $I \subset J$, <u>then</u> $V_I[g] \leq V_J[g]$

Thm 1. $\quad g \in BV(I) \implies g$ is bdd. in I.

(Pf) $\quad g(x) = g(a) + g(x) - g(a)$

$\implies |g(x)| \leq |g(a)| + |g(x) - g(a)|$

$$\leq |g(a)| + V_a^x[g]$$

$$\leq |g(a)| + V_a^b[g] < \infty. \quad /\!/\!/$$

One can also show that $BV(I)$ is a Banach space.

Thm 2. $\quad g, h \in BV(I) \implies gh \in BV(I).$
$\quad g, h \in BV(I), |h(x)| \geq \exists m > 0$
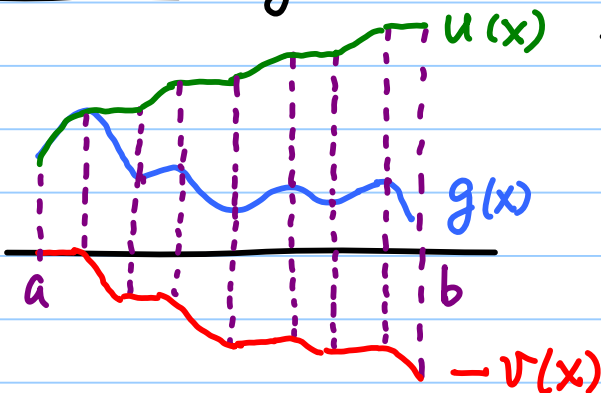$$\implies g/h \in BV(I).$$

Thm 3. $\quad \forall c \in (a, b), \quad g \in BV[a, b]$
$$\iff g \in BV[a, c] \ \& \ g \in BV[c, b].$$

Moreover, $V_a^b[g] = V_a^c[g] + V_c^b[g].$

Remark : This can be generalized to
$$a < c_1 < c_2 < \cdots < c_n < b.$$

Thm. 4 $\quad g \in BV(I) \iff g$ can be written as the difference of two nondecreasing fcns.

say, $u(x) - v(x)$

M. Taibleson's Thm (1967)   1 page paper!

If $f \in BV[0,1]$, $f(x) \sim \sum_{-\infty}^{\infty} \alpha_k e^{2\pi i k x}$,

then $\alpha_k = O(1/k)$ as $k \to \infty$.

(Pf) Use the fact: $k \neq 0$

$$\int_{j/|k|}^{\frac{j+1}{|k|}} e^{-2\pi i k x} dx = 0, \quad j = 0, 1, \cdots, |k|,$$

$$\odot \hookrightarrow = \frac{e^{-2\pi i (j+1)\frac{k}{|k|}} - e^{-2\pi i j \frac{k}{|k|}}}{-2\pi i k}$$

$$= \frac{1}{-2\pi i k}\left(e^{\mp 2\pi i (j+1)} - e^{\mp 2\pi i j}\right) = 0. \quad //$$

Now, fix k, and let $a_j := \frac{j}{|k|}$, $j = 0, 1, \cdots; |k|$.
Then define

$$g(x) := \sum_{j=0}^{|k|-1} f(a_j) \chi_{[a_j, a_{j+1}]}(x) \qquad \begin{array}{l}\text{a step fcn}\\ \text{approx } f.\end{array}$$

The k th Fourier coeff. of g.

Then, $\alpha_k[g] = \int_0^1 g(x) e^{-2\pi i k x} dx$

$$= \int_0^1 \sum_{j=0}^{|k|-1} f(a_j) \chi_{[a_j, a_{j+1}]}(x) e^{-2\pi i k x} dx$$

$$= \sum_{j=0}^{|k|-1} f(a_j) \int_{a_j}^{a_{j+1}} e^{-2\pi i k x} dx = 0.$$

$$\alpha_k[f] = \int_0^1 f(x) e^{-2\pi i k x} dx$$

$$|\alpha_k[f]| = |\alpha_k[f] - \alpha_k[g]| = |\alpha_k[f-g]|$$
$$= \left| \int_0^1 (f(x) - g(x)) e^{-2\pi i k x} dx \right|$$

$$\leq \sum_{j=0}^{|k|-1} \int_{a_j}^{a_{j+1}} |f(x) - f(a_j)| \, dx$$

$$\leq \sum_{j=0}^{|k|-1} V_{a_j}^{a_{j+1}}[f] \underbrace{(a_{j+1} - a_j)}_{= 1/|k|}$$

Thm 3 $\longrightarrow$

$$\overset{\downarrow}{=} \frac{1}{|k|} \underbrace{V_0^1[f]}_{\text{"const.}} = O(1/|k|). \quad /\!/\!/$$

Thm (N.S.-J.F. Remy, 2006)

Let f be a 1-periodic fcn and $f \in C^m(\mathbb{R})$. Furthermore, let us assume that $f^{(m+1)}$ exists and in $BV[0,1]$. Then its Fourier coeff.
$\alpha_k[f] = \hat{f}(k)$ decays as $O(|k|^{-m-2})$, where $m = 0, 1, \cdots$.

(Pf) Use $\begin{cases} \text{the periodicity, i.e., } f^{(\ell)}(0) = f^{(\ell)}(1), \ell = 0, \cdots; m. \\ \text{integration by parts!} \end{cases}$

$$\hat{f}(k) = \int_0^1 f(x) e^{-2\pi i k x} \, dx$$

$$= \frac{e^{-2\pi i k x}}{-2\pi i k} f(x) \Big|_0^1 + \frac{1}{2\pi i k} \int_0^1 f'(x) e^{-2\pi i k x} \, dx$$

$$= \frac{e^{-2\pi i k x}}{-(2\pi i k)^2} f'(x) \Big|_0^1 + \frac{1}{(2\pi i k)^2} \int_0^1 f''(x) e^{-2\pi i k x} \, dx$$

$$= \cdots = \frac{e^{-2\pi i k x}}{-(2\pi i k)^{m+1}} f^{(m)}(x) \Big|_0^1 + \frac{1}{(2\pi i k)^{m+1}} \int_0^1 f^{(m+1)}(x) e^{-2\pi i k x} \, dx$$

By assumption, $f^{(m+1)} \in BV[0,1]$. So, can use the Taibleson Thm to get:

$$|\hat{f}(k)| \leq V_0^1[f^{(m+1)}] (2\pi)^{-m-1} \cdot |k|^{-m-2}. \quad /\!/\!/$$

☆ Fourier Series on Intervals II

Suppose your fcn is defined on $[0, \frac{A}{2}]$ instead of $[-\frac{A}{2}, \frac{A}{2}]$.

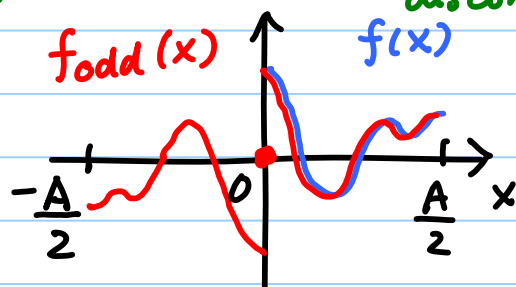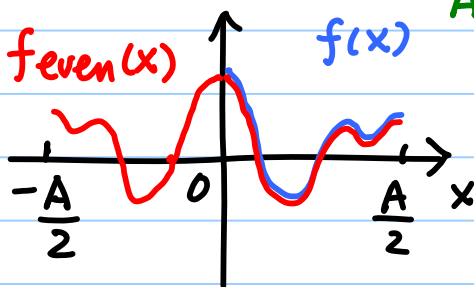$\Rightarrow \exists$ two ways to make it an A-periodic fcn:

(1) Even extension

$$f_{even}(x) = \begin{cases} f(x) & \text{if } x \in [0, \frac{A}{2}] \\ f(-x) & \text{if } x \in [-\frac{A}{2}, 0] \end{cases}$$

$f \in C[0, \frac{A}{2}] \Rightarrow f_{even} \in C(\mathbb{R})$
A-periodic

$f_{even}(x)$ $f(x)$

$-\frac{A}{2}$ 0 $\frac{A}{2}$ x

(2) Odd extension

$$f_{odd}(x) = \begin{cases} f(x) & \text{if } x \in (0, \frac{A}{2}] \\ 0 & \text{if } x = 0 \\ -f(-x) & \text{if } x \in [-\frac{A}{2}, 0) \end{cases}$$

$f \in C[0, \frac{A}{2}] \Rightarrow f_{odd}:$
discontinuous

$f_{odd}(x)$ $f(x)$

$-\frac{A}{2}$ 0 $\frac{A}{2}$ x

Then their Fourier series expansions get simpler. But before computing them, let's review the relationship between the Fourier coefficients $\{\alpha_k\}$ w.r.t. the ONB $\{\frac{1}{\sqrt{A}} e^{2\pi i k x /A}\}$ and $\{a_k, b_k\}$ w.r.t. the ONB $\{\frac{1}{\sqrt{A}}\} \cup \{\sqrt{\frac{2}{A}} \cos(\frac{2\pi k}{A} x)\} \cup \{\sqrt{\frac{2}{A}} \sin(\frac{2\pi k}{A} x)\}$.

Let $g(x)$ be an A-periodic L^2 fcn.

$$g(x) \sim \frac{1}{\sqrt{A}} \sum_{k \in \mathbb{Z}} \alpha_k e^{2\pi i k x /A}, \quad \alpha_k = \frac{1}{\sqrt{A}} \int_{-A/2}^{A/2} g(x) e^{-2\pi i k x /A} \, dx$$

Then $\frac{1}{\sqrt{A}} \sum_{k \in \mathbb{Z}} \alpha_k e^{2\pi i k x /A}$

$$= \frac{1}{\sqrt{A}} \left[\alpha_0 + \sum_{1}^{\infty} (\alpha_k + \alpha_{-k}) \cos\left(\frac{2\pi k x}{A}\right) + i(\alpha_k - \alpha_{-k}) \sin\left(\frac{2\pi k x}{A}\right) \right]$$

$$= \frac{\alpha_0}{\sqrt{A}} + \sum_1^\infty \boxed{\frac{\alpha_k + \alpha_{-k}}{\sqrt{2}}} \sqrt{\frac{2}{A}} \cos\left(\frac{2\pi k x}{A}\right) + \boxed{\frac{\alpha_k - \alpha_{-k}}{\sqrt{2}} i} \sqrt{\frac{2}{A}} \sin\left(\frac{2\pi k x}{A}\right)]$$

$\underbrace{\qquad}_{a_k} \qquad \underbrace{\qquad}_{b_k}$

If we want, we can write a_0 instead of α_0.

<u>Remark:</u> In many books, the Fourier series is often defined in $[-\pi, \pi)$ and written as

$$\sum_{k=-\infty}^\infty c_k e^{ikx} = \frac{a_0}{2} + \sum_{k=1}^\infty \left(a_k \cos kx + b_k \sin kx\right),$$

where $a_0 = 2c_0$, $a_k = c_k + c_{-k}$, $b_k = i(c_k - c_{-k})$

$\qquad\qquad\qquad\qquad k \geq 1$

But $\{e^{ikx}\}$ is an <u>orthogonal</u> basis of $L^2[-\pi, \pi]$, but <u>not normalized</u>. To make it an ONB, one needs the factor $\frac{1}{\sqrt{2\pi}}$.

The same can be said for the orthogonal basis $\{1\} \cup \{\cos kx\} \cup \{\sin kx\}$.

The ONB is $\left\{\frac{1}{\sqrt{2\pi}}\right\} \cup \left\{\frac{1}{\sqrt{\pi}} \cos kx\right\} \cup \left\{\frac{1}{\sqrt{\pi}} \sin kx\right\}$

i.e., $A = 2\pi$.

Compare this notation with mine in this lecture, which is the <u>orthonormalized</u> version:

$$\sum_{k=-\infty}^\infty \alpha_k \frac{1}{\sqrt{A}} e^{2\pi i k x / A} = \frac{1}{\sqrt{A}} a_0 + \sqrt{\frac{2}{A}} \sum_{k=1}^\infty \left(a_k \cos\left(\frac{2\pi k}{A} x\right) \right.$$
$$\left. + b_k \sin\left(\frac{2\pi k}{A} x\right)\right),$$

where $a_0 = \alpha_0$, $a_k = \frac{\alpha_k + \alpha_{-k}}{\sqrt{2}}$, $b_k = \frac{\alpha_k - \alpha_{-k}}{\sqrt{2}} i$

$\qquad\qquad k \geq 1$.

Now, let's go back to the Fourier series
expansion of f_{even} & f_{odd}.

$$\alpha_k[f_{even}] = \frac{1}{\sqrt{A}} \int_{-A/2}^{A/2} f_{even}(x) e^{-2\pi ikx/A} dx$$

$$= \frac{2}{\sqrt{A}} \int_{0}^{A/2} f(x) \cos\left(\frac{2\pi kx}{A}\right) dx$$

$$= \alpha_{-k}[f_{even}] \qquad \text{\color{green}{thanks to the evenness of } } \cos\theta.$$

Recall the relationship:

$$a_0 = \alpha_0, \quad a_k = \frac{\alpha_k + \alpha_{-k}}{\sqrt{2}}, \quad b_k = \frac{\alpha_k - \alpha_{-k}}{\sqrt{2}} i;$$

Hence, in this case of f_{even}, $a_0 = \alpha_0[f_{even}]$,
$$a_k = \sqrt{2}\,\alpha_k[f_{even}], \quad b_k \equiv 0, \quad k \geq 1.$$

In other words, f_{even} can be written as
the Fourier cosine series :

$$f_{even}(x) \sim \frac{1}{\sqrt{A}} a_0 + \sum_{k=1}^{\infty} a_k \sqrt{\frac{2}{A}} \cos\left(\frac{2\pi kx}{A}\right)$$

where $a_0 = \frac{2}{\sqrt{A}} \int_{0}^{A/2} f(x) dx = \alpha_0[f_{even}]$,

$$a_k = 2\sqrt{\frac{2}{A}} \int_{0}^{A/2} f(x) \cos\left(\frac{2\pi kx}{A}\right) dx$$

$$= \sqrt{2}\,\alpha_k[f_{even}].$$

Similarly, for $f_{odd}(x)$,

$$\alpha_k[f_{odd}] = \frac{1}{\sqrt{A}} \int_{-A/2}^{A/2} f_{odd}(x) e^{-2\pi i k x/A} dx$$

$$= \frac{-2i}{\sqrt{A}} \int_0^{A/2} f(x) \sin\left(\frac{2k\pi x}{A}\right) dx$$

$$= -\alpha_{-k}[f_{odd}] \quad \text{due to the oddness of } \sin\theta.$$

$$\Rightarrow \quad a_k = \frac{\alpha_k + \alpha_{-k}}{\sqrt{2}} \equiv 0, \quad k \in \mathbb{N}. \quad \left.\begin{array}{c} \text{No} \\ \text{cosines} \end{array}\right.$$

$$a_0 = \alpha_0 = \frac{1}{\sqrt{A}} \int_{-A/2}^{A/2} f_{odd}(x) dx = 0.$$

$$b_k = \frac{\alpha_k - \alpha_{-k}}{\sqrt{2}} i = \sqrt{2} \, i \, \alpha_k[f_{odd}]$$

So, $$f_{odd}(x) \sim \sum_{k=1}^{\infty} b_k \sqrt{\frac{2}{A}} \sin\left(\frac{2\pi k x}{A}\right)$$

$$b_k = 2\sqrt{\frac{2}{A}} \int_0^{A/2} f(x) \sin\left(\frac{2\pi k x}{A}\right) dx.$$

So, if a fcn is given on $[0, \frac{A}{2}]$, say $f \in C[0, \frac{A}{2}]$
$\exists$ three ways to extend it to a periodic fcn

$O(1/k)$ (1) Brute force periodization with period $\frac{A}{2}$.
$O(1/k^2)$ (2) Even extension followed by A-periodization.
$O(1/k)$ (3) Odd 〃 〃 〃

(2) is the **best among these 3** in terms of the decay of the Fourier coeff's.

However, $\exists$ an even better way!

�— The Lanczos Method (1938)

Suppose $f \in C^{2m}[0,1]$, but $f(0) \neq f(1)$ no match at $x = 0, 1$.
also assume $f^{(2m+1)} \in BV[0,1]$. $m = 1, 2, \cdots$.

Lanczos's idea:
    decompose $\quad f(x) = u(x) + v(x)$

where $u(x) = $ a polynomial of degree $2m-1$.
s.t. $\begin{cases} u^{(2k)}(0) = f^{(2k)}(0) \\ u^{(2k)}(1) = f^{(2k)}(1) \end{cases}$, $k = 0, 1, \cdots, m-1$.

Then, consider the **odd** extension of v.

$\Rightarrow v \in C^{2m-1}(\mathbb{R})$, $\quad v^{(k)}(0) = v^{(k)}(1) = 0$
$$k = 0, 1, 2, \cdots, 2m-1.$$

and the Fourier sine coefficients of
$v(x)$ (with period 2) decay as
$$b_k = O\left(|k|^{-2m-1}\right)$$

e.g., $m = 1$ gives us $b_k = O(1/k^3)$.
    and $u(x)$ is a straight line
    connecting $(0, f(0))$ & $(1, f(1))$.

Remarks:

(1) $f = u + v = $ (an algebraic poly) + (a trig. poly).
can avoid both the **Runge** & **Gibbs** phenomena!

(2) NS-J.F.Remy (2006) generalized this to $\mathbb{R}^d$.

Lagrange Polynomial Interpolation at 11 equispaced points to 1/(1+25x^2)
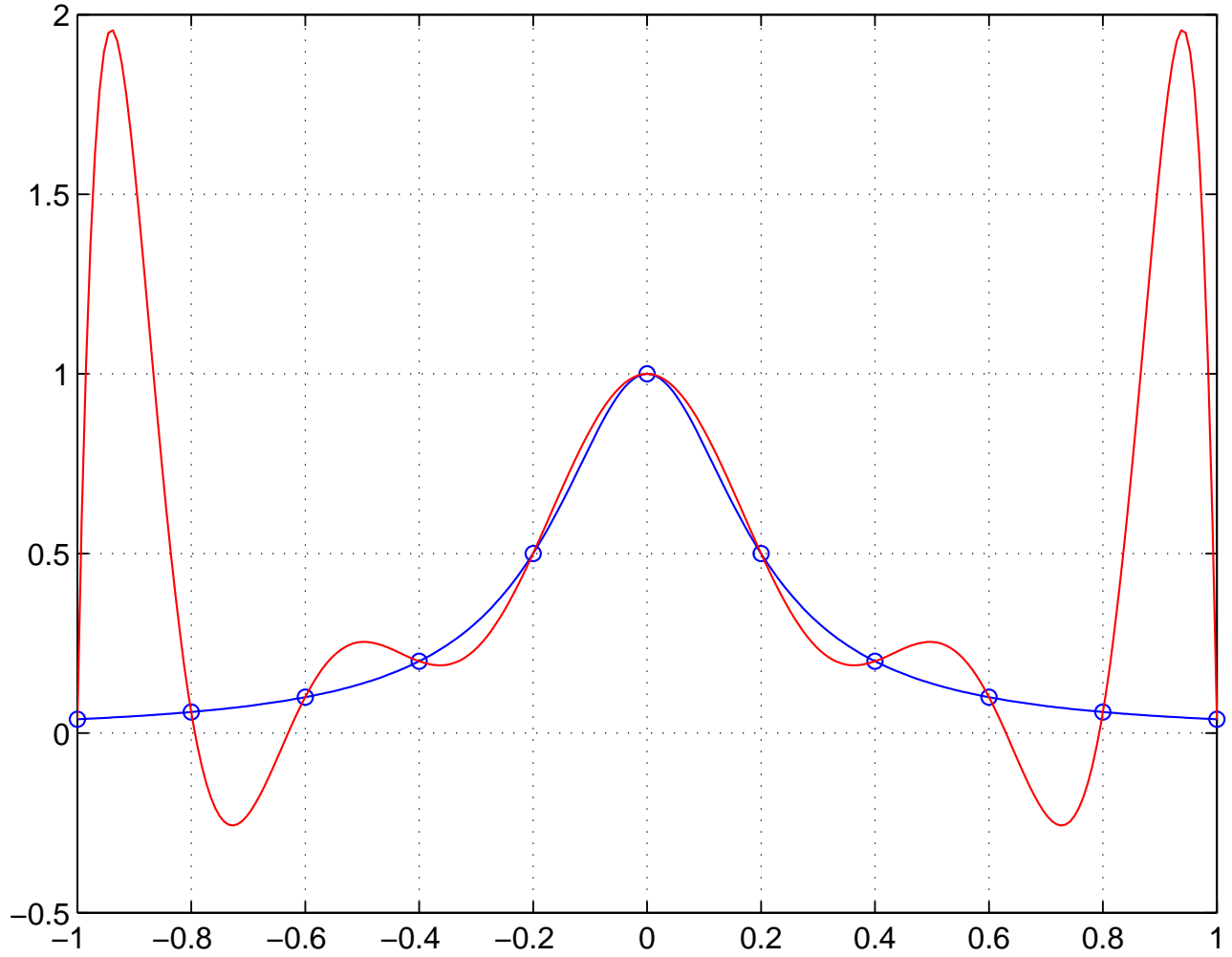
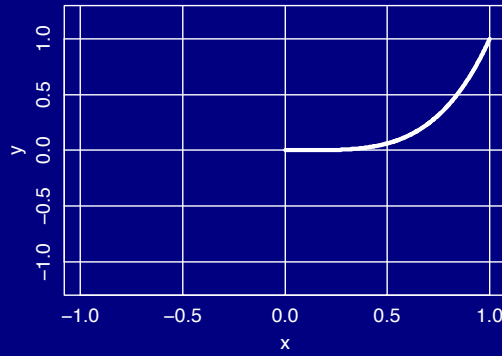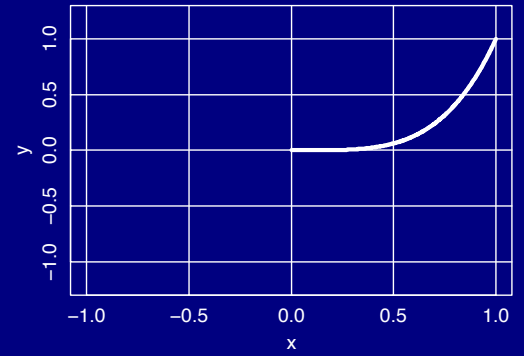Lagrange Polynomial Interpolation at 11 random points to 1/(1+25x^2)
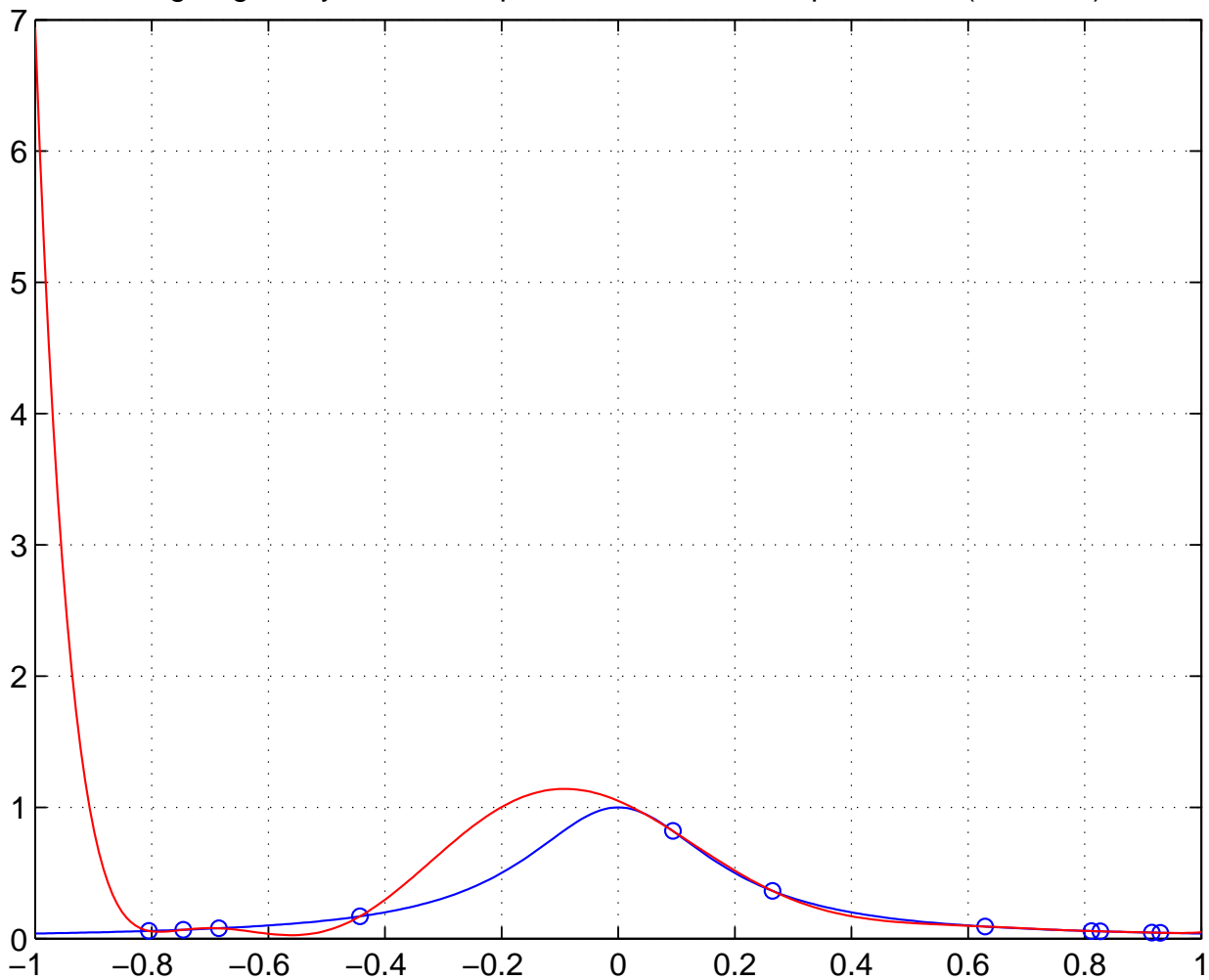
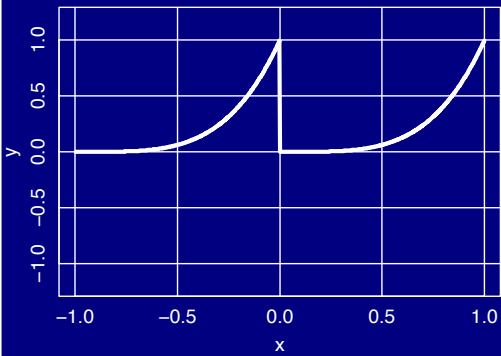Gibbs phenomenon
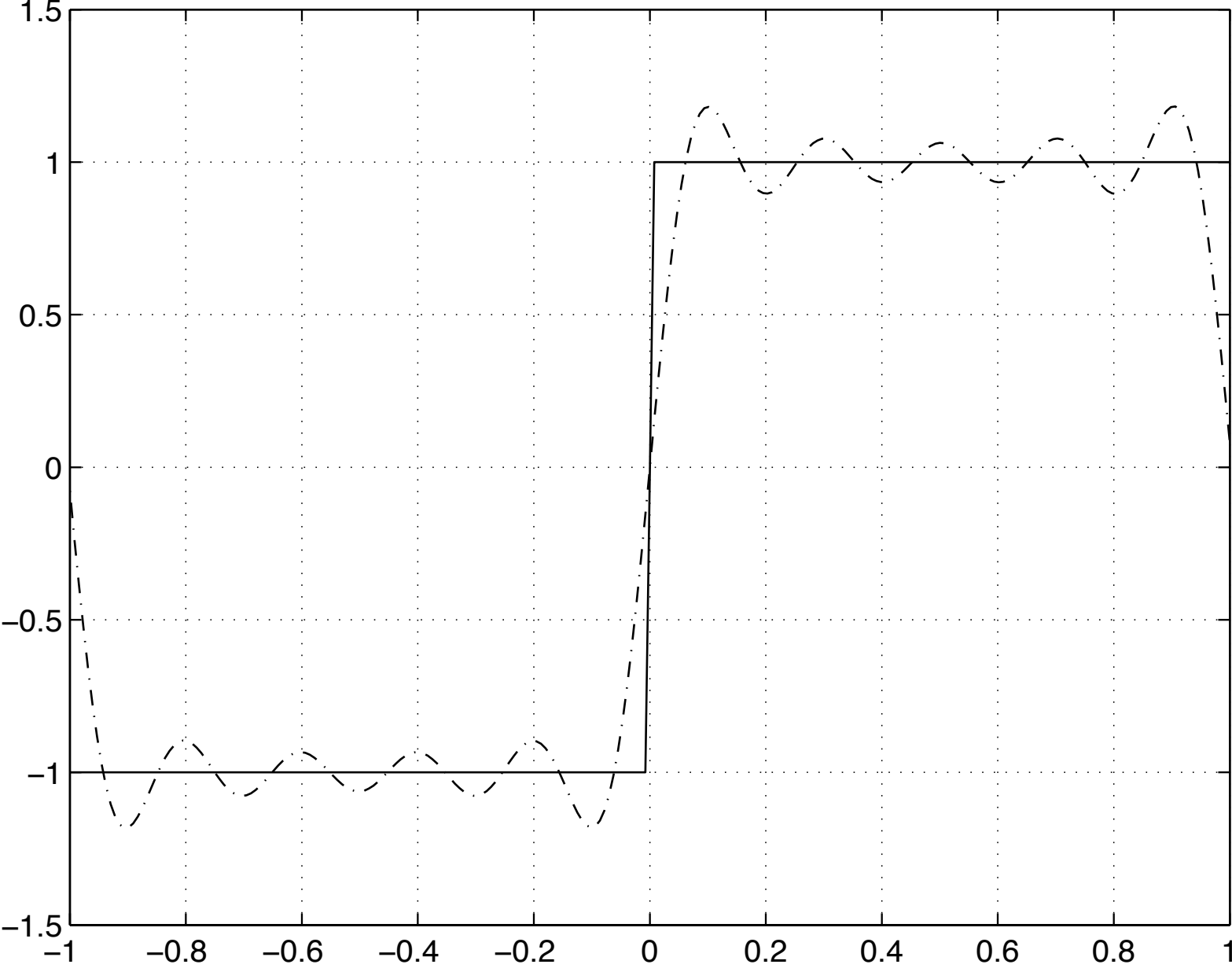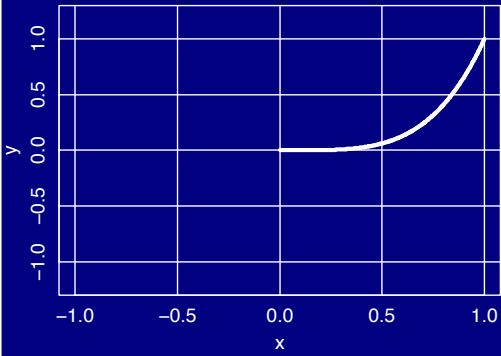
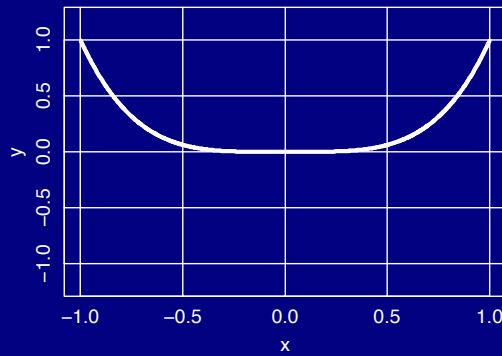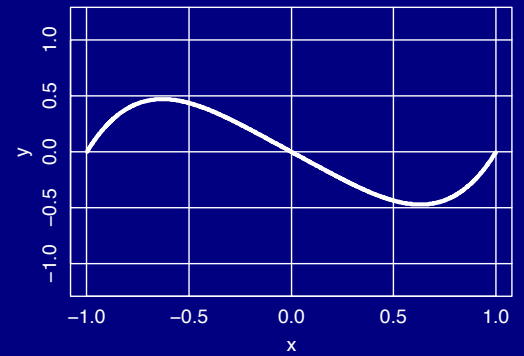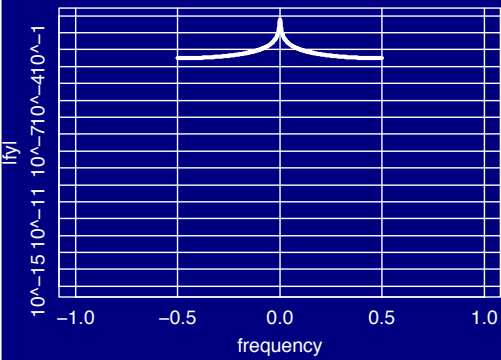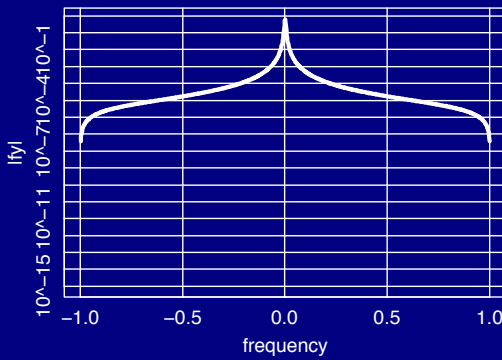| Original Signal Supported on [0,1] | Original Signal Supported on [0,1] | Original Signal Supported on [0,1] |
| After Periodization | After Even Reflection | After Lin Removal+Odd Reflect |
| DFT Coefficients | DCT Coefficients | LLST Coefficients |