

Lecture 9: to Discrete Cosine/Sine Transforms

★ Fourier Series, Boundary Value Problems, and the Sturm-Liouville Theory

It is important to note that the Fourier basis fns $\left\{ \frac{1}{\sqrt{A}} e^{2\pi i k x / A} \right\}_{k \in \mathbb{Z}}$ are **eigenfunctions** of the following BVP of the 2nd order ODE:

1D Laplacian eigenval. Problem!

$$\begin{cases} -u''(x) = \lambda u(x) & x \in \left[-\frac{A}{2}, \frac{A}{2}\right] \\ u(-A/2) = u(A/2) \\ u'(-A/2) = u'(A/2) \end{cases} \text{ periodic bdry. cond.!$$

This is one example of the so-called **regular Sturm-Liouville Problem**.

λ is the eigenvalue, in fact

$$\lambda = \lambda_k = (2\pi k / A)^2, \text{ and the corresponding eigenfcn is } \varphi_k(x) = \frac{1}{\sqrt{A}} e^{2\pi i k x / A}$$

Def. A **regular Sturm-Liouville problem** on the interval $I = [a, b]$ is specified by the following data:

(i) A formally self-adjoint differential operator \mathcal{L} defined as

$$\mathcal{L} u(x) := \frac{1}{w(x)} \left\{ - (p(x) u'(x))' + q(x) u(x) \right\}, \quad \forall x \in I.$$

where $p \in C^1(I)$, $q, w \in C(I)$, $p > 0$, $w > 0$, $q \in \mathbb{R} \quad \forall x \in I.$

(ii) A set of self-adjoint bdry. cond.'s
 $B_1(u) = 0$ & $B_2(u) = 0$ for L .

The objective of a regular SL problem is to find all solutions of the following BVP:

$$\begin{cases} Lu = \lambda u \\ B_1(u) = B_2(u) = 0 \end{cases}$$

\Rightarrow Solutions exist for specific λ , i.e., eigenvalues of such rSLP.

Define $L_w^2[a, b] := \{f \mid \|f\|_{2,w} < \infty\}$
 where $\|f\|_{2,w}^2 = \|f\|_w^2 := \int_a^b |f(x)|^2 w(x) dx$

Define the **weighted inner product**:

$$\langle f, g \rangle_w := \int_a^b f(x) \overline{g(x)} w(x) dx.$$

Going back to the operator L .

$\forall f, g \in L_w^2[a, b]$,

$$\langle Lf, g \rangle_w = \langle f, L^*g \rangle_w$$

Int. by parts \Downarrow

$$= \langle f, Lg \rangle_w + \left[-p(f' \bar{g} - f \bar{g}') \right]_a^b$$

if L is formally self-adjoint.

If $B_j(f) = B_j(g) = 0$, $j=1,2$, $f, g \in L^2_w[a,b]$,
lead to $[-p(f' \bar{g} - f \bar{g}')]_a^b = 0$, then

these bdry. cond.'s are said to be self-adjoint, and together with the formally self-adjoint operator \mathcal{L} , we have:

$$\langle \mathcal{L}f, g \rangle_w = \langle f, \mathcal{L}g \rangle_w$$

In this case, the problem is called self-adjoint.

The most important thm's here are:

Thm For every rSLP, the following holds:

- (a) All eigenvalues are real;
- (b) Eigenfunctions corresponding to distinct eigenvalues are orthogonal w.r.t. $\langle \cdot, \cdot \rangle_w$;
- (c) The eigenspace (i.e., the subspace spanned by those eigenfcn's belonging to an eigenval.) for any eigenvalue λ is at most 2 dim.

If the bdry. cond. is separated, it is always 1 dim.

Thm For every rSLP, \exists an ONB $\{\varphi_n\}_{n \in \mathbb{N}}$ of $L^2_w[a,b]$ s.t. $\{\varphi_n\}$ are eigenfcn's.

If λ_n : the corresp. eigenval. to φ_n , then $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$. Moreover if

$f \in C^2[a,b]$, $B_1(f) = B_2(f) = 0$, then

$$\sum_{n=1}^N \langle f, \varphi_n \rangle_w \varphi_n \rightarrow f \text{ uniformly as } N \rightarrow \infty.$$

So, $\left\{ \begin{array}{l} -u'' = \lambda u, \\ u(-A/2) = u(A/2) \\ u'(-A/2) = u'(A/2) \end{array} \right\}$ is one of the simplest SLP!

Remark: A **singular** SLP is an SLP with
 (i) $p(a) = 0$ or $p(b) = 0$.
 in addition $w(a) = 0$ or $+\infty$ or $w(b) = 0$ or $+\infty$;
 or (ii) $a = -\infty$ or $b = +\infty$.

Almost all of classical orthogonal polynomials are generated by SLP's with specific B.C. & weight fcn's.

	$-p$	q	w	λ_n	$[a, b]$
Legendre poly.	$1-x^2$	0	1	$n(n+1)$	$[-1, 1]$
Chebyshev poly.	$\sqrt{1-x^2}$	0	$\frac{1}{\sqrt{1-x^2}}$	n^2	$[-1, 1]$
Hermite poly.	e^{-x^2}	0	e^{-x^2}	$2n$	$(-\infty, \infty)$
$\alpha > -1$ Laguerre poly.	$x^{\alpha+1} e^{-x}$	0	$x^\alpha e^{-x}$	n	$[0, \infty)$
Prolate Spheroidal wave fcn's	$1-x^2$	$-cx^2$	1	λ	$[-1, 1]$
\vdots	\vdots	\vdots	\vdots	$(0 < \lambda < 1)$	\vdots

In 2D and higher, the problem becomes more intricate, of course.

The simplest version is that of **Laplacian eigenvalue problem**:

$$-\Delta u = \lambda u \quad \text{in } \Omega = \text{a domain in } \mathbb{R}^d$$

$$\left. \begin{array}{l} \text{Dirichlet} \\ \text{Neumann} \end{array} \right\} \text{ or } \begin{cases} u = 0 \\ \partial_\nu u = 0 \end{cases} \quad \text{on } \partial\Omega \text{ (a bdy of } \Omega \text{)}.$$

★ Fourier Sine & Cosine Series

... come out naturally as eigenfcn's on the simple r SLP with the following B.C.'s:

$$-u'' = \lambda u, \quad x \in \left[-\frac{A}{2}, \frac{A}{2}\right].$$

$$\text{i.e., } p(x) \equiv 1, \quad q(x) \equiv 0, \quad w(x) \equiv 1.$$

$$\text{Dirichlet B.C.: } u\left(-\frac{A}{2}\right) = u\left(\frac{A}{2}\right) = 0 \Rightarrow \left\{ \sqrt{\frac{2}{A}} \sin \frac{2\pi k}{A} x \right\}_{k=1}^{\infty}$$

$$\text{Neumann B.C.: } u'\left(-\frac{A}{2}\right) = u'\left(\frac{A}{2}\right) = 0 \Rightarrow \left\{ \frac{1}{\sqrt{A}} \right\} \cup \left\{ \sqrt{\frac{2}{A}} \cos \frac{2\pi k}{A} x \right\}_{k=1}^{\infty}$$

But clearly \exists other possibilities, e.g.,

$$u\left(-\frac{A}{2}\right) = u'\left(\frac{A}{2}\right) = 0 \Rightarrow \left\{ \sqrt{\frac{2}{A}} \sin \frac{2\pi(k+\frac{1}{2})}{A} x \right\}_{k=0}^{\infty}$$

$$u'\left(-\frac{A}{2}\right) = u\left(\frac{A}{2}\right) = 0 \Rightarrow \left\{ \sqrt{\frac{2}{A}} \cos \frac{2\pi(k+\frac{1}{2})}{A} x \right\}_{k=0}^{\infty}$$

Discretization gives us further intricacies!

★ Discrete Sine & Cosine Transforms

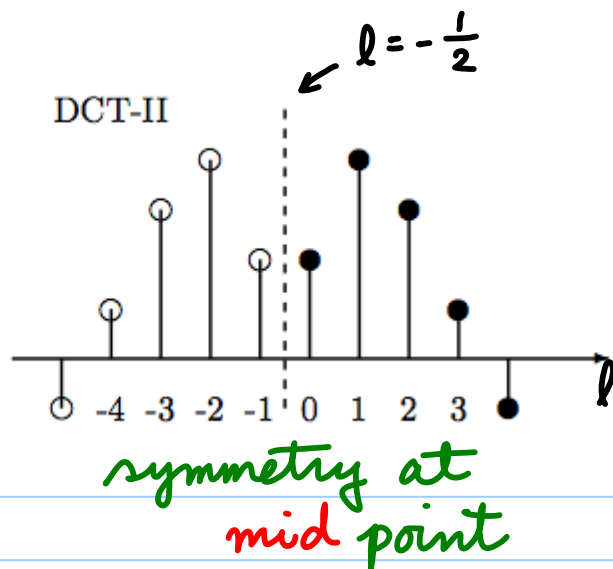
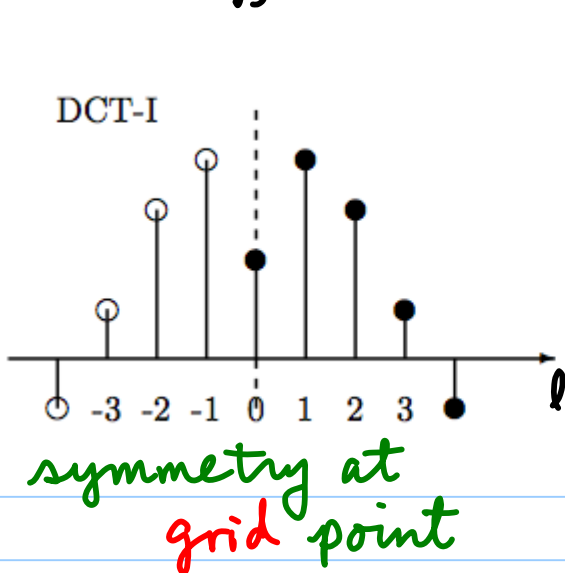
Define

$$\begin{cases} F_S^I[k] := \sum_{l=1}^{N-1} f[l] \sin \left(\frac{\pi k l}{N} \right) : \text{DST-I} \\ F_C^I[k] := \sum_{l=0}^{N-1} f[l] \cos \left(\frac{\pi k l}{N} \right) : \text{DCT-I} \end{cases}$$

not $\frac{2\pi k l}{N}$ Type

These can be computed via normal FFT (of length $2N$) by appropriately extending the original sequence.

\exists 4 different types of DSTs & DCTs with different B.C.'s.



For convenience, let's define the following weight vector for orthogonality:

$$b[l] := \begin{cases} 0 & \text{if } l < 0 \text{ or } l > N; \\ 1/\sqrt{2} & \text{if } l = 0 \text{ or } l = N; \\ 1 & \text{if } 1 \leq l \leq N-1. \end{cases}$$

Now we can define the following transf. matrices:

$$\left\{ \begin{array}{l} \text{DCT-I: } C_{N+1}^{\text{I}} \in \mathbb{R}^{(N+1) \times (N+1)}, \quad C_{N+1}^{\text{I}}[k, l] = b[k]b[l] \sqrt{\frac{2}{N}} \cos \frac{\pi k l}{N} \\ \text{DCT-II: } C_N^{\text{II}} \in \mathbb{R}^{N \times N}, \quad C_N^{\text{II}}[k, l] = b[k] \sqrt{\frac{2}{N}} \cos \frac{\pi k (l + \frac{1}{2})}{N} \\ \text{DCT-III: } C_N^{\text{III}} \in \mathbb{R}^{N \times N}, \quad C_N^{\text{III}}[k, l] = b[l] \sqrt{\frac{2}{N}} \cos \frac{\pi (k + \frac{1}{2}) l}{N} \\ \text{DCT-IV: } C_N^{\text{IV}} \in \mathbb{R}^{N \times N}, \quad C_N^{\text{IV}}[k, l] = \sqrt{\frac{2}{N}} \cos \frac{\pi (k + \frac{1}{2}) (l + \frac{1}{2})}{N} \end{array} \right.$$

For DCT-I,

$k, l = 0, 1, \dots, N$

For others, $k, l = 0, 1, \dots, N-1$.

k : frequency index

l : space (or time) index.

$$\left\{ \begin{array}{l} \text{DST-I: } S_{N-1}^{\text{I}} \in \mathbb{R}^{(N-1) \times (N-1)}, \quad S_{N-1}^{\text{I}}[k, l] = \sqrt{\frac{2}{N}} \sin \frac{\pi k l}{N} \\ \text{DST-II: } S_N^{\text{II}} \in \mathbb{R}^{N \times N}, \quad S_N^{\text{II}}[k, l] = b[k+1] \sqrt{\frac{2}{N}} \sin \frac{\pi (k+1)(l+\frac{1}{2})}{N} \\ \text{DST-III: } S_N^{\text{III}} \in \mathbb{R}^{N \times N}, \quad S_N^{\text{III}}[k, l] = b[l+1] \sqrt{\frac{2}{N}} \sin \frac{\pi (k+\frac{1}{2})(l+1)}{N} \\ \text{DST-IV: } S_N^{\text{IV}} \in \mathbb{R}^{N \times N}, \quad S_N^{\text{IV}}[k, l] = \sqrt{\frac{2}{N}} \sin \frac{\pi (k+\frac{1}{2})(l+\frac{1}{2})}{N} \end{array} \right.$$

For DST-I, $k, l = 1, \dots, N-1$

For others, $k, l = 0, 1, \dots, N-1$.

Remarks:

(1) In the JPEG image compression standard, the 2D version of DCT-II is used on patches of size 8×8 pixels via the tensor product of 1D DCT-II.

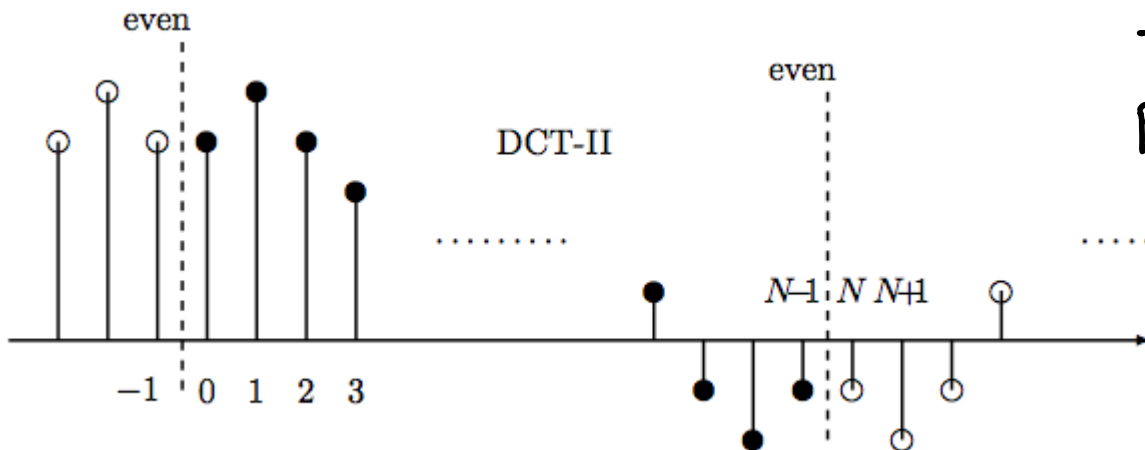
(2) The MATLAB function "dct" is the DCT-II, and is included in the Signal Processing Toolbox.

(3) The MATLAB function "dst" is the DST-I (unnormalized version), and is included in the PDE Toolbox.

★ Comments on the B.C.'s

	<u>left endpt.</u>		<u>right endpt</u>
DCT-I	grid pt, Neumann		grid pt, Neumann
II	mid pt, Neumann		mid pt, Neumann
III	grid pt, Neumann		grid pt, Dirichlet
IV	mid pt, Neumann		mid pt, Dirichlet
DST-I	grid pt, Dirichlet		grid pt, Dirichlet
II	mid pt, Dirichlet		mid pt, Dirichlet
III	grid pt, Dirichlet		grid pt, Neumann
IV	mid pt, Dirichlet		mid pt, Neumann

★ DCT-II



Then
periodized
with period
2N

Define

$$\tilde{f}[l] := \begin{cases} f[l] & \text{if } l = 0, 1, \dots, N-1. \\ f[2N-l-1] & \text{if } l = N, \dots, 2N-1. \end{cases}$$

Then consider

$$\begin{aligned} D_{2N}\{\tilde{f}\}[k] &= \sum_{l=0}^{2N-1} \tilde{f}[l] \omega_{2N}^{-kl} \\ &= \sum_{l=0}^{N-1} f[l] \omega_{2N}^{-kl} + \sum_{l=N}^{2N-1} f[2N-l-1] \omega_{2N}^{-kl} \\ &= \sum_{l=0}^{N-1} f[l] \omega_{2N}^{-kl} + \sum_{m=N-1}^0 f[m] \omega_{2N}^{-k(2N-1-m)} \quad \leftarrow 2N-l-1=m \\ &= \sum_{l=0}^{N-1} f[l] \omega_{2N}^{-kl} + \sum_{m=0}^{N-1} f[m] \omega_{2N}^{km} \cdot \omega_{2N}^k \quad \leftarrow \omega_{2N}^{-k \cdot 2N} = 1 \\ &= \sum_{l=0}^{N-1} f[l] \left(\omega_{2N}^{-kl} + \omega_{2N}^{\left(\frac{k}{2} + \frac{k}{2}\right) kl} \omega_{2N}^{kl} \right) \\ &= \omega_{2N}^{k/2} \sum_{l=0}^{N-1} f[l] \left(\omega_{2N}^{-k(l+\frac{1}{2})} + \omega_{2N}^{k(l+\frac{1}{2})} \right) \\ &= 2 e^{\frac{\pi i k}{N}} \sum_{l=0}^{N-1} f[l] \cos \frac{\pi k(l+\frac{1}{2})}{N} \end{aligned}$$

$$\Rightarrow \frac{1}{\sqrt{2N}} D_{2N}\{\tilde{f}\}[k] = \sqrt{\frac{2}{N}} e^{\frac{\pi i k}{N}} \sum_{l=0}^{N-1} f[l] \cos \frac{\pi k(l+\frac{1}{2})}{N}$$

Viewing samples at half integers on the x-axis in the DFT set up eliminates this phase factor.

- The inverse transform to DCT-II is DCT-III!