# Lecture 13: Wavelet Transforms

Note Title 2/15/2014

## * Problems of STFT & Gabor systems

- If a window is too large (wide), then cannot localize around sharp transitions in an input signal.
- If a window is too small (narrow), then cannot detect low freq. oscillations.
- The Balian-Low Thm: $\nexists$ "nice" Gabor ONBs

## * Key idea of wavelets:

Use translations and dilations of a single fcn to analyze a given signal at different resolutions.

**Def.** A wavelet is a fcn $\psi \in L^2(\mathbb{R})$ s.t.

often called a "mother" wavelet

- $\int_{-\infty}^{\infty} \psi(x)\,dx = 0$;
- Normalized to have $\|\psi\| = 1$; and
- Centered around $x = 0$.

Let's generate a family of TF atoms:

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{x-b}{a}\right)$$

$$= \tau_b\,\delta_a\,\psi \qquad a > 0,\ b \in \mathbb{R}$$

Note $\|\psi_{a,b}\|_2 = 1$.

The **wavelet transform** of $f \in L^2(\mathbb{R})$ is defined as

$$Wf(a,b) = W_\psi f(a,b) := \langle f, \psi_{a,b} \rangle$$

often called the "continuous" wavelet transf.

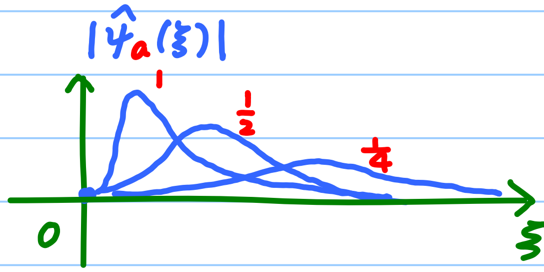
$$= \int_{-\infty}^{\infty} f(x) \frac{1}{\sqrt{a}} \overline{\psi\left(\frac{x-b}{a}\right)}\, dx$$

Can be viewed as a linear filtering:

$$\int_{-\infty}^{\infty} f(x) \frac{1}{\sqrt{a}} \overline{\psi\left(\frac{x-b}{a}\right)}\, dx = f \underset{\text{conv.}}{*} \tilde{\psi}_a(b)$$

$$\underbrace{\tilde{\psi}_a(x)}_{\text{mirror of } \psi_a(x)} := \frac{1}{\sqrt{a}} \psi\left(\frac{-x}{a}\right) \xrightarrow{\mathcal{F}} \hat{\tilde{\psi}}_a(\xi) = \sqrt{a}\, \overline{\hat{\psi}(a\xi)}$$

$$= \overline{S_{1/a} \hat{\psi}(\xi)}$$




## Types of wavelets:

- Real wavelets $\rightarrow$ Good for **edges**
- Analytic (or complex) wavelets $\rightarrow$ can detect **phases**

For the time being, let's focus on real wavelets.

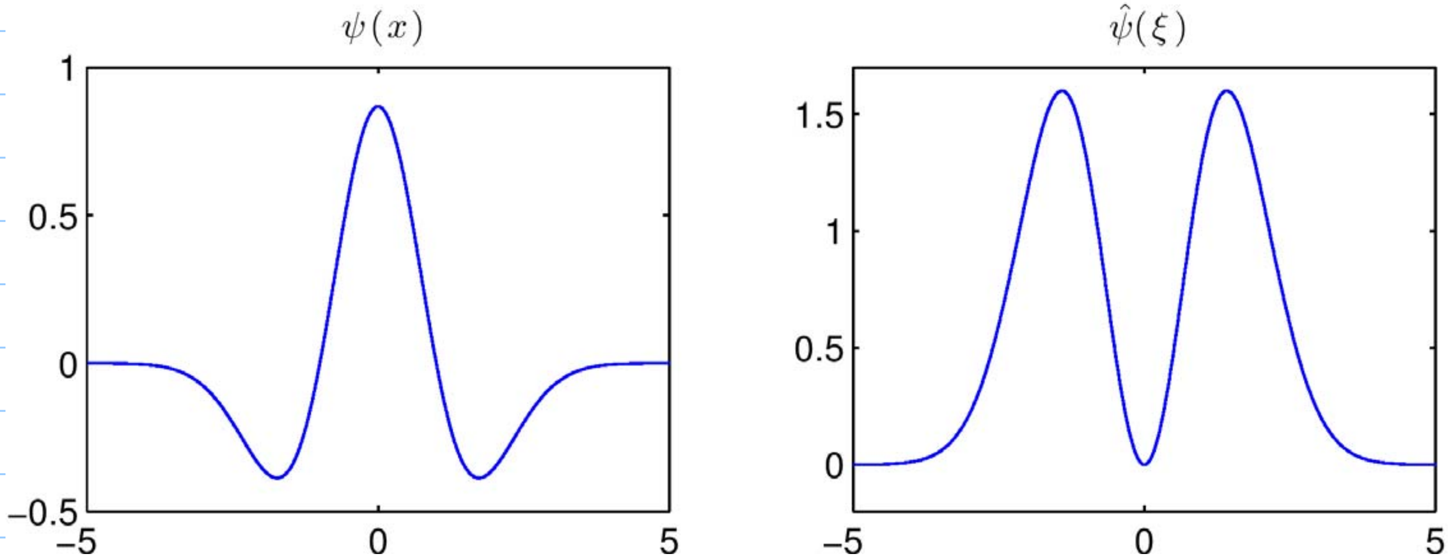
Example: Mexican hat fcn or a.k.a. Laplacian of Gaussian (LoG)

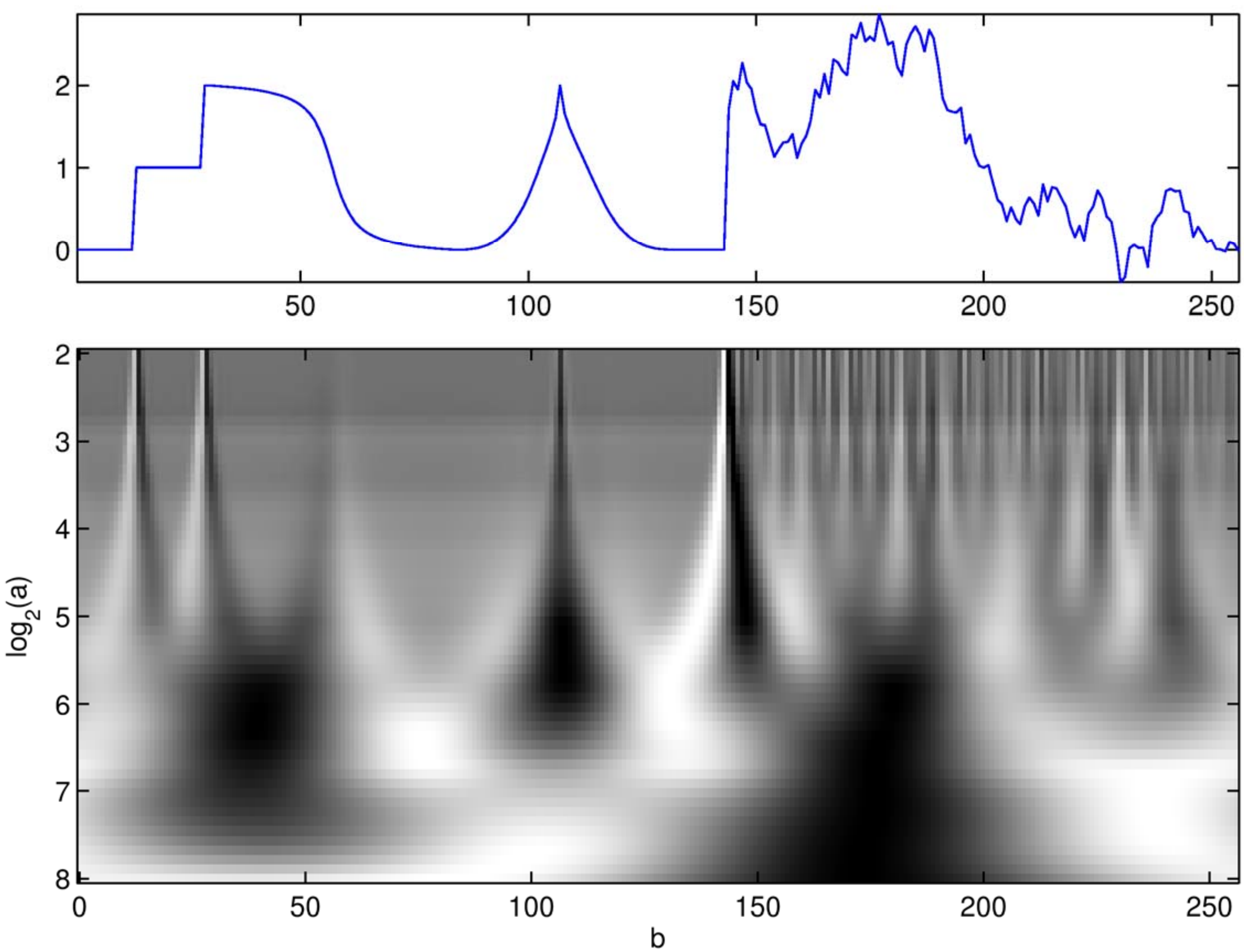
$$\begin{cases} \psi(x) = \dfrac{2}{\pi^{\frac{1}{4}}\sqrt{3\sigma}} \left(1 - \dfrac{x^2}{\sigma^2}\right) e^{-x^2/2\sigma^2} \\ \hat{\psi}(\xi) = 8\sqrt{\dfrac{2}{3}}\, \pi^{\frac{9}{4}} \sigma^{\frac{5}{2}} \xi^2 e^{-2\pi^2\sigma^2\xi^2} \end{cases}$$

$\hat{\psi}(0) = 0$, $\quad \hat{\psi}(\xi) \sim \xi^2$ around $\xi = 0$

$\rightarrow$ approx. to $\frac{d^2}{dx^2}$

often called a **pseudo differential op.**


$\psi(x)$
$\hat{\psi}(\xi)$


$\log_2(a)$
b

## Inverse Wavelet Transform

1964 1984

**Thm (Calderón - Grossmann - Morlet)**

Let $\psi \in L^2(\mathbb{R})$, $\psi \in \mathbb{R}$ s.t.

$$C_\psi := \int_0^\infty \frac{|\hat{\psi}(\xi)|^2}{\xi} d\xi < +\infty$$

Then any $f \in L^2(\mathbb{R})$ satisfies

$$(*) \quad f(x) = \frac{1}{C_\psi} \int_0^\infty \int_{-\infty}^{+\infty} Wf(a,b)\, \psi_{a,b}(x)\, db\, \frac{da}{a^2}$$

and $$\|f\|_2^2 = \frac{1}{C_\psi} \int_0^\infty \int_{-\infty}^{\infty} |Wf(a,b)|^2\, db\, \frac{da}{a^2}.$$

(Pf) $Wf(a,b) = f * \tilde{\psi}_a(b)$

$$\text{RHS of } (*) = \frac{1}{C_\psi} \int_0^\infty \left(Wf(a,\cdot) * \psi_{a,\cdot}\right)(x) \frac{da}{a^2}$$

$$= \frac{1}{C_\psi} \int_0^\infty \left(f * \tilde{\psi}_a * \psi_a\right)(x) \frac{da}{a^2}$$

$$\xrightarrow{\mathcal{F}} = \frac{1}{C_\psi} \int_0^\infty \hat{f}(\xi) \sqrt{a}\, \overline{\hat{\psi}(a\xi)} \sqrt{a}\, \hat{\psi}(a\xi) \frac{da}{a^2}$$

$$= \frac{\hat{f}(\xi)}{C_\psi} \int_0^\infty \frac{|\hat{\psi}(a\xi)|^2}{a} da$$

$a\xi = \eta$

$$= \frac{\hat{f}(\xi)}{C_\psi} \underbrace{\int_0^\infty \frac{|\hat{\psi}(\eta)|^2}{\eta} d\eta}_{= C_\psi} = \hat{f}(\xi) \quad ///$$

$C_\psi < +\infty$ is called the admissibility condition, $(*)$ is called Calderón's reproducing formula.

$$f(x) = \frac{1}{C_\psi} \int_0^\infty f * \tilde{\psi}_a * \psi_a(x) \frac{da}{a^2}$$

↳ also called the resolution of identity.

To guarantee $C_\psi < \infty$, we need
$\hat{\psi}(0) = 0 \iff \underbrace{\int_{-\infty}^{\infty} \psi(x)\, dx = 0}$

so, $\psi$ must be oscillatory with $\pm$ values

also need decay on $\psi$
e.g., $\int_{-\infty}^{\infty} (1+|x|)\, \psi(x)\, dx < \infty$.

## ☆ Reproducing Kernel

CWT = a redundant representation

$$(**) \quad Wf(a,b) = \int_{-\infty}^{\infty} \underbrace{\left( \frac{1}{C_\psi} \int_0^\infty \int_{-\infty}^{\infty} Wf(a',b')\, \psi_{a',b'}(x)\, db' \frac{da'}{a'^2} \right)}_{= f(x)} \overline{\psi_{a,b}(x)}\, dx$$

$$= \frac{1}{C_\psi} \int_0^\infty \int_{-\infty}^{\infty} K(a,a',b,b')\, Wf(a',b')\, db' \frac{da'}{a'^2}$$

where $K(a,a',b,b') := \langle \psi_{a,b}, \psi_{a',b'} \rangle$

↪ measuring the correlation between $\psi_{a,b}$ & $\psi_{a',b'}$

If $K(a,a',b,b') = \delta(a-a')\,\delta(b-b')$
then no redundancy!

Prop. a function $\Phi(a,b) \in L^2(\mathbb{R}_+ \times \mathbb{R})$ is a wavelet transform of some $f \in L^2(\mathbb{R})$
$\iff$ $\Phi(a,b)$ satisfies $(**)$.

## ☆ Scaling Function (Father Wavelet)

Reconstruction formula requires all values of scales $0 < a < +\infty$

If we only know $Wf(a,b)$ for $a < a_0$,
Then we need complementary info. for $a > a_0$ provided by the scaling function (father wavelet):

$$|\hat{\phi}(\xi)|^2 := \int_1^\infty |\hat{\psi}(a\xi)|^2 \frac{da}{a}$$

$$= \int_\xi^\infty \frac{|\hat{\psi}(\eta)|^2}{\eta} d\eta$$

The phase of $\phi$ can be arbitiary chosen.

- $\lim_{\xi \to 0} |\hat{\phi}(\xi)|^2 = C_\psi$
- $\|\phi\|_2 = 1$ ← Exercise, use the def.

So, the low freq. approx. of $f$ at scale $a$ can be written as

$$Lf(a,x) := \langle f, \overbrace{\delta_a \phi}^{=\phi_a} \rangle = f * \tilde{\phi}_a(x)$$

$$\Rightarrow \quad f(x) = \frac{1}{C_\psi} \int_0^{a_0} (Wf(a,\cdot) * \psi_a)(x) \frac{da}{a^2}$$

$$+ \underbrace{\frac{1}{C_\psi a_0} (Lf(a_0,\cdot) * \phi_{a_0})(x)}_{\text{complementary info}}$$

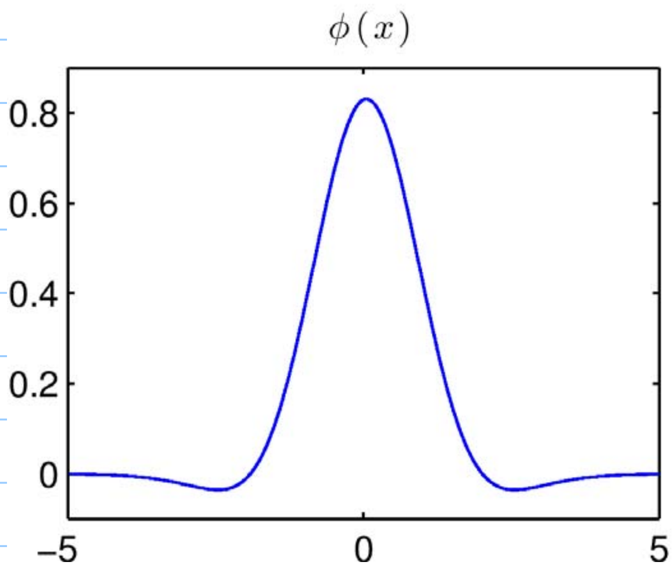
Ex. $$\phi(x) = \frac{2}{\pi^{\frac{1}{4}}\sqrt{3\sigma}}\left(1-\frac{x^2}{\sigma^2}\right)e^{-x^2/2\sigma^2}$$

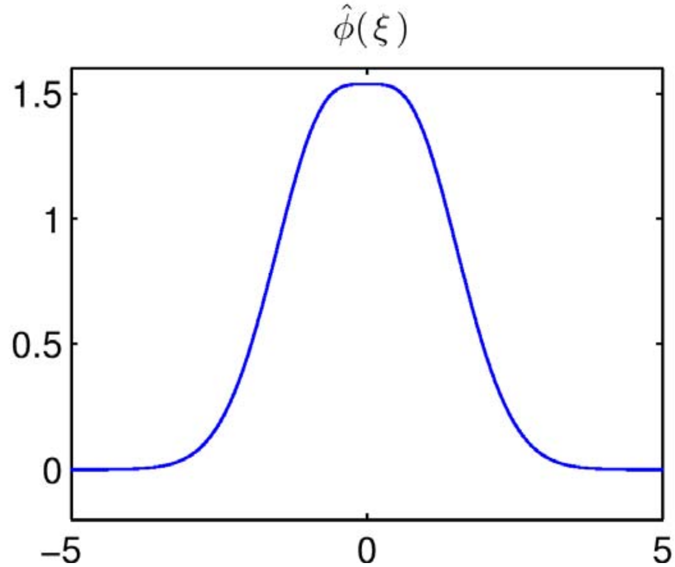
$$\hat{\phi}(\xi) = 8\sqrt{\frac{2}{3}}\,\pi^{\frac{9}{4}}\sigma^{\frac{5}{2}}\xi^2 e^{-2\pi^2\sigma^2\xi^2}$$

$$\Rightarrow |\hat{\phi}(\xi)|^2 = \frac{4\sigma}{3\sqrt{\pi}}\left(1+4\pi^2\sigma^2\xi^2\right)e^{-4\pi^2\sigma^2\xi^2}$$

$$\Rightarrow \hat{\phi}(\xi) = 2\sqrt{\frac{\sigma}{3\sqrt{\pi}}}\sqrt{1+4\pi^2\sigma^2\xi^2}\;e^{-2\pi^2\sigma^2\xi^2}$$

↳ choose a simple phase factor.



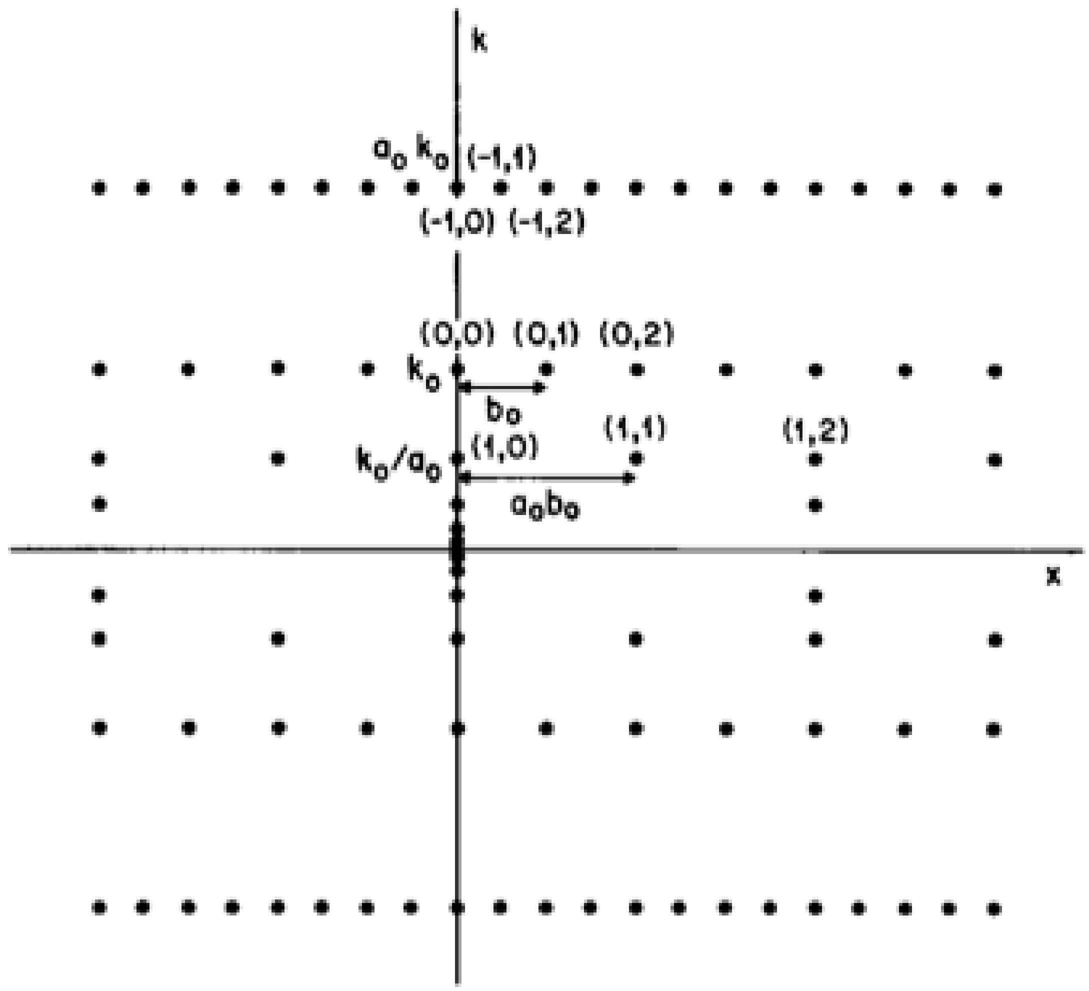
## ☆ Discrete Wavelet Transforms

How to sample $Wf(a,b)$ ??

⇒ another great insight by J. Morlet

"regular hyperbolic grid"

$$(a,b) = (a_0^m, n a_0^m b_0), \quad m, n \in \mathbb{Z}$$



a bit technical

**Thm (Regular sampling thm, Daubechies '90)**

Let $\psi$ be a real-valued $L^2$-function. For fixed $a_0, b_0$, define

$$\psi_{m,n}(x) := a_0^{-m/2}\psi(a_0^{-m}x - nb_0), \quad m, n \in \mathbb{Z}$$
$$= \frac{1}{\sqrt{a_0^m}}\psi\left(\frac{x - na_0^m b_0}{a_0^m}\right)$$

(1) <u>If</u> $\{\psi_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ is a frame of $L^2(\mathbb{R})$ with the frame bounds $A, B$, <u>then</u> we must have

$$A \le \frac{1}{b_0}\sum_{-\infty}^{\infty}|\hat{\psi}(a_0^m\xi)|^2 \le B \text{ for } \xi \in \mathbb{R} \text{ a.e.}$$

In particular, $\psi$ satisfies the admissibility cond.

$$C_\psi = \int_0^\infty |\hat{\psi}(\xi)|^2 \frac{d\xi}{\xi} < +\infty$$

(2) <u>If</u>, for some $\varepsilon > 0$, $\psi$ satisfies $|x|^{\frac{1}{2}+\varepsilon}\psi \in L^2$, $|\xi|^\varepsilon\hat{\psi} \in L^2$ and $\int \psi(x)\,dx = 0$, <u>then</u> $\psi$ satisfies:

$$(*)\begin{cases} \operatorname{ess\,inf} \sum_{m\in\mathbb{Z}} |\hat{\psi}(a_0^m\xi)|^2 > 0 \\ \operatorname{ess\,sup} \sum_{m\in\mathbb{Z}} |\hat{\psi}(a_0^m\xi)|^2 < +\infty \end{cases} \text{ for any } a_0 \text{ close enough to } 1.$$

(i.e., $\exists\, \alpha = \alpha(\psi) > 1$ s.t. $(*)$ is satisfied $\forall a_0 \in (1, \alpha)$.)

Moreover, <u>if</u> $b_0$ is close enough to $0$ (i.e., $\exists\, \beta = \beta(a_0, \psi)$ s.t. $(*)$ is satisfied $\forall\, b_0 \in (0, \beta)$), <u>then</u> $\{\psi_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ constitute a frame!

Ex. $\psi(x) =$ the Mexican hat fcn
$a_0 = 2, \; b_0 = 1/4.$

$\Rightarrow \{\psi_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ forms a frame
(called a wavelet frame)
$A = 13.09, \; B = 14.18$ i.e., almost tight !

<u>Dual Frame</u> : Wavelet frame operator $U$ commutes with dilations $\delta_{a_0^m}$,
but not with translations $\tau_{na_0^m b_0}$

$\Rightarrow$ dual frame $\left\{(U^*U)^{-1}\tau_{na_0^m b_0}\,\delta_{a_0^m}\psi\right\}_{(m,n)\in\mathbb{Z}^2}$
is in general not a wavelet system
(unlike the Gabor frame).