Lecture 15: Wavelet Bases

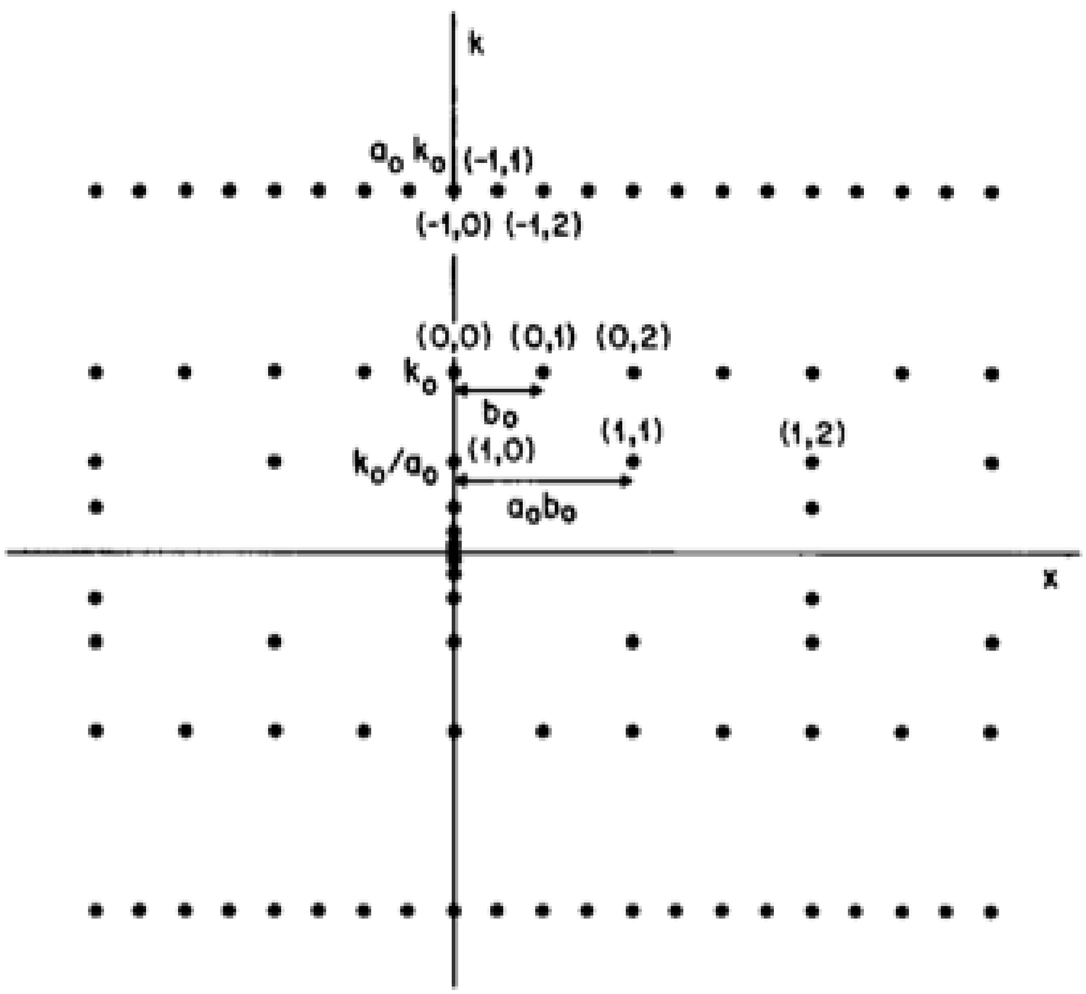
Note Title 2/24/2014

- Recall the sampling lattice pattern ("regular hyperbolic grid") to generate the discrete wavelet transf. from the continuous wavelet transf. $Wf(a,b)$.

$$(a,b) = (a_0^m, n a_0^m b_0), \quad m, n \in \mathbb{Z}$$

$$\psi_{m,n}(x) := a_0^{-m/2} \psi(a_0^{-m} x - n b_0), \quad m, n \in \mathbb{Z}$$

$$= \frac{1}{\sqrt{a_0^m}} \psi\left(\frac{x - n a_0^m b_0}{a_0^m}\right)$$



The most common/popular choice of (a,b) is $a = 2^j$, $b = 2^j k$ i.e., $a_0 = 2$, $b_0 = 1$.

$$\Rightarrow \psi_{jk}(x) = \frac{1}{\sqrt{2^j}} \psi\left(\frac{x - 2^j k}{2^j}\right) = 2^{-j/2} \psi(2^{-j} x - k), \quad (j,k) \in \mathbb{Z}^2.$$

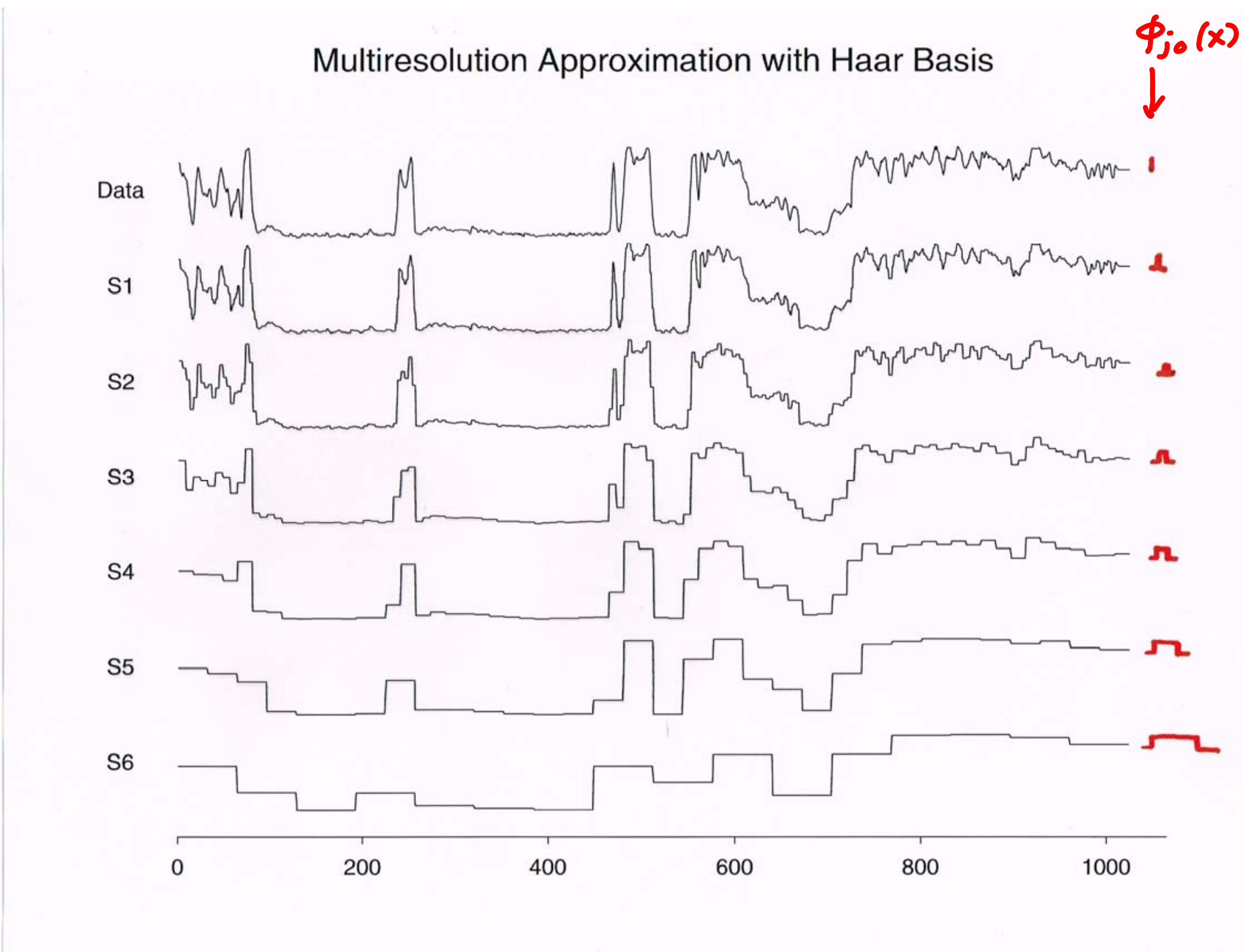
☆ Multiresolution Approximation

The parameter 2^{-j} represents resolution whereas 2^{j} represents scale, i.e.,

high resolution $\Leftrightarrow$ fine (small) scale
low " coarse (large) "

For example, for a fixed j, consider the translated & dilated versions of a father wavelet : $\{\phi_{jk}\}_{k\in\mathbb{Z}}$, and the approximation (or projection) of $f \in L^2(\mathbb{R})$ as follows :

$$f_j(x) = \sum_{k\in\mathbb{Z}} \langle f, \phi_{jk} \rangle \phi_{jk}(x)$$



View the approximation of f at resolution 2^{-j} as an orthogonal projection P_j (or P_{V_j}) of f onto a subspace:

$$V_j := \overline{\operatorname{span}\{\phi_{jk}, k \in \mathbb{Z}\}} \subset L^2(\mathbb{R})$$

__Def.__ (Multiresolution Approximation)

a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ is called a multiresolution approximation (MRA) if the following six properties are satisfied:

(1) $\forall (j,k) \in \mathbb{Z}^2$, $f(x) \in V_j \iff f(x - 2^j k) \in V_j$

$j\uparrow$: high res. (2) $\forall j \in \mathbb{Z}$, $V_{j+1} \subset V_j$

$j\downarrow$: low res. (3) $\forall j \in \mathbb{Z}$, $f(x) \in V_j \iff f\left(\frac{x}{2}\right) \in V_{j+1}$

(4) $\lim_{j\to\infty} V_j = \bigcap_{j=-\infty}^{\infty} V_j = \{0\}$

(5) $\lim_{j\to-\infty} V_j = \overline{\bigcup_{j=-\infty}^{\infty} V_j} = L^2(\mathbb{R})$

(6) $\exists\, \theta \in V_0$, s.t. $\{\theta(x-k)\}_{k\in\mathbb{Z}}$ form a Riesz basis for V_0.

linearly indep. frame (i.e., a basis) with $0 < A \leq 1 \leq B$ (if $\|\theta\| = 1$).

__Remark:__

$$P_{V_j} f \in V_j, \quad \underbrace{P_{V_j}^2 = P_{V_j}}_{\text{proj}}, \quad \underbrace{P_{V_j}^* = P_{V_j}}_{\text{orthogonal proj}}$$

$$\lim_{j\to\infty} \|P_{V_j} f\| = 0, \quad \lim_{j\to-\infty} \|f - P_{V_j} f\| = 0.$$

• Now $\{\theta(x-k)\}_{k\in\mathbb{Z}}$ is a Riesz basis for V_0

$\Longleftrightarrow$ $\forall f \in V_0$, $\exists! \{a_k\} \in \ell^2(\mathbb{Z})$ s.t.

$$f(x) = \sum_{k\in\mathbb{Z}} a_k \theta(x-k) \quad \text{with}$$

$$A\|f\|^2 \leq \sum |a_k|^2 \leq B\|f\|^2, \quad \exists A \gneq 0, \ \exists B > 0$$

Then, we can adapt this $\{\theta(x-k)\}_{k\in\mathbb{Z}}$ by dilation for V_j, i.e.,

$\{2^{-j/2}\theta(2^{-j}x-k)\}_{k\in\mathbb{Z}}$ is a Riesz basis for V_j

with the same frame bounds A, B.

<u>Prop.</u> a family $\{\theta(x-k)\}_{k\in\mathbb{Z}}$ is a Riesz basis for $V_0 = \overline{\text{span}\{\theta(x-k)\}}_{k\in\mathbb{Z}}$ $\Longleftrightarrow$ $\exists B \geq A > 0$ s.t.

$$\forall \xi \in [-\tfrac{1}{2}, \tfrac{1}{2}], \quad \frac{1}{B} \leq \sum_{k\in\mathbb{Z}} |\hat{\theta}(\xi-k)|^2 \leq \frac{1}{A}$$

(Pf) ($\Longleftarrow$) Take any $f \in V_0$.

$$f(x) = \sum_{k\in\mathbb{Z}} a_k \theta(x-k)$$

$$\Big\downarrow \mathcal{F}$$

$$\hat{f}(\xi) = \hat{a}(\xi)\hat{\theta}(\xi), \quad \hat{a}(\xi) := \sum_{k\in\mathbb{Z}} a_k e^{-2\pi i k\xi}$$

$\hookrightarrow$ 1-periodic!

Now, $\|f\|^2 = \|\hat{f}\|^2 = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi$

$$= \int_{-1/2}^{1/2} \sum_{\ell\in\mathbb{Z}} |\hat{a}(\xi+\ell)|^2 |\hat{\theta}(\xi+\ell)|^2 d\xi$$

$\hat{a}(\xi)$: 1-periodic

$$= \int_{-1/2}^{1/2} |\hat{a}(\xi)|^2 \sum_{\ell\in\mathbb{Z}} |\hat{\theta}(\xi+\ell)|^2 d\xi$$

Hence, together with the assumption, we have

$$\frac{1}{B}\int_{-1/2}^{1/2}|\hat{a}(\xi)|^2 d\xi \leq \|f\|^2 \leq \frac{1}{A}\int_{-1/2}^{1/2}|\hat{a}(\xi)|^2 d\xi$$

But $\displaystyle\int_{-1/2}^{1/2}|\hat{a}(\xi)|^2 d\xi = \sum_{k\in\mathbb{Z}}|a_k|^2$ via Parseval

Hence, $\displaystyle A\|f\|^2 \leq \sum_{k\in\mathbb{Z}}|a_k|^2 \leq B\|f\|^2$ ✓

($\Rightarrow$) If $\{\theta(x-k)\}_{k\in\mathbb{Z}}$ is a Riesz basis for V_0, then $\exists\, B \geq A > 0$ s.t.

$$A\|f\|^2 \leq \sum_{k\in\mathbb{Z}}|a_k|^2 \leq B\|f\|^2$$

$$\sum_{k\in\mathbb{Z}}|a_k|^2 = \int_{-1/2}^{1/2}|\hat{a}(\xi)|^2 d\xi$$

Reversing the argument, we get

$$\frac{1}{B} \leq \sum_{k\in\mathbb{Z}}|\hat{\theta}(\xi-k)|^2 \leq \frac{1}{A}, \quad \forall \xi \in [-\tfrac{1}{2},\tfrac{1}{2}].$$

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<u>Example 1</u>. $\theta(x) = \chi_{[0,1)}(x)$

$\Rightarrow$ Piecewise constant approximation

$$\theta_{j,k}(x) = 2^{-j/2}\chi_{[k2^j,(k+1)2^j)}(x)$$

$\{\theta_{j,k}\}_{k\in\mathbb{Z}}$ form an ONB for V_j.

<u>Example 2</u>. $\displaystyle\theta(x) = \mathrm{sinc}(x) = \frac{\sin\pi x}{\pi x}$

$$\Leftrightarrow \hat{\theta}(\xi) = \chi_{[-1/2,1/2)}(\xi)$$

This is also called the Shannon scaling fcn.

$$\theta_{j,k}(x) = 2^{-j/2}\mathrm{sinc}(2^{-j}x-k) \Leftrightarrow \hat{\theta}_{j,k}(\xi) = 2^{j/2}e^{-2\pi i k 2^j\xi}\chi_{[-2^{-j-1},2^{-j-1})}(\xi)$$

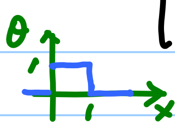
$\{\theta_{j,k}\}_{k\in\mathbb{Z}}$ form an ONB for $V_j = \underbrace{BL_{2^{-j}}(\mathbb{R})}_{\text{Band limited fcn with bandwidth } 2^{-j}}$

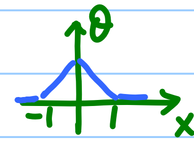
Example 3. Spline of order m
(a.k.a. cardinal B-spline)

$$\theta(x) = \chi_{[0,1]} \underbrace{* \chi_{[0,1]} * \cdots *}_{m \text{ times convolution}} \chi_{[0,1]}(x)$$

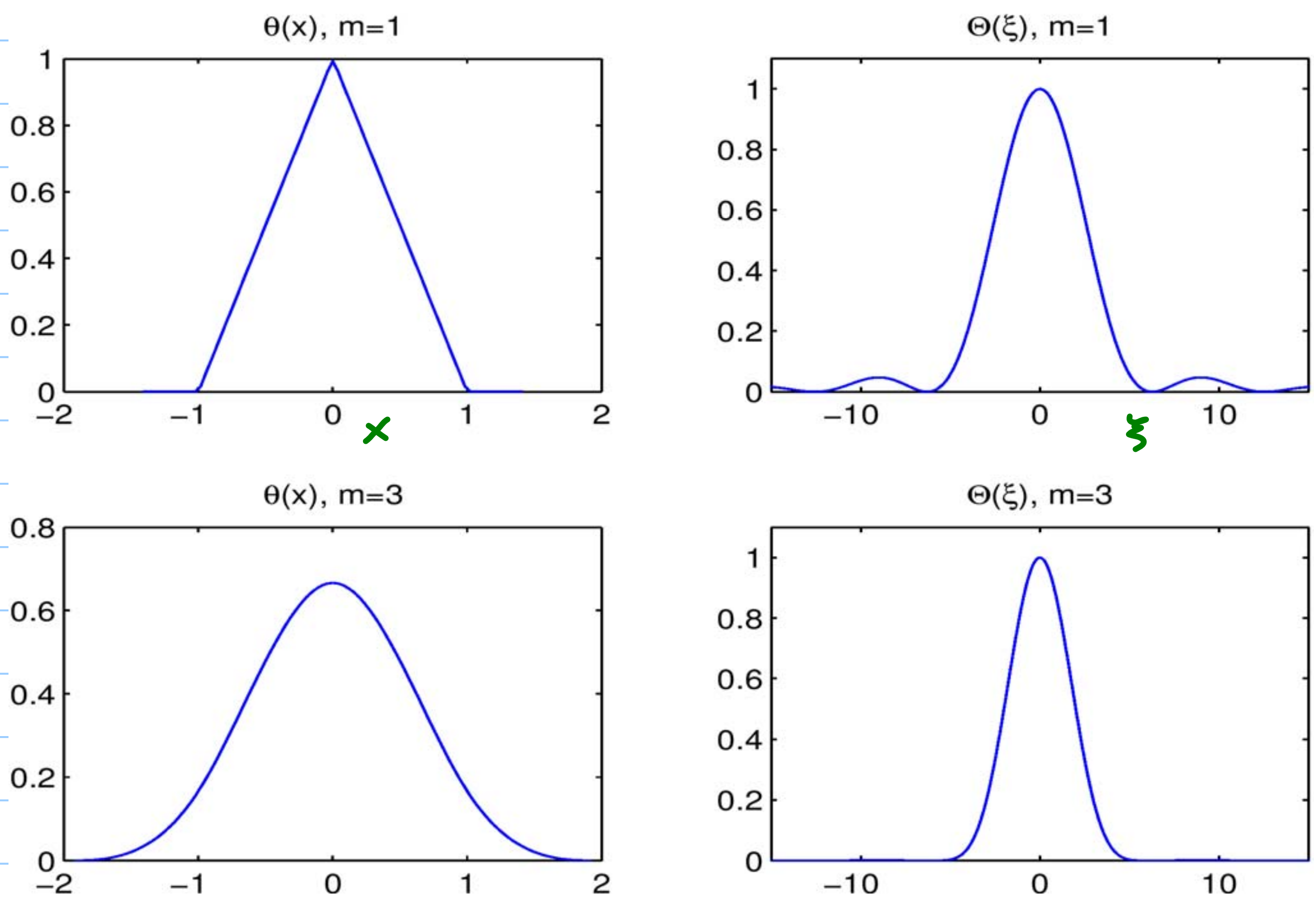
with centering at $x = \begin{cases} 0 & \text{if } m = \text{odd} \\ 1/2 & \text{if } m = \text{even} \end{cases}$

$\mathcal{F}$

m = 0 : Box spline 

m = 1 : linear spline (hat fcn) 

$$\hat{\theta}(\xi) = \left(\frac{\sin \pi\xi}{\pi\xi}\right)^{m+1} \underbrace{e^{-i\varepsilon\pi\xi}}_{\text{half-shift factor}} \qquad \varepsilon = \begin{cases} 0 & m = \text{odd} \\ 1 & m = \text{even} \end{cases}$$



☆ Scaling Functions (Father Wavelets)

Can we construct an ONB from the Riesz basis $\{\theta(x-k)\}_{k\in\mathbb{Z}}$ for V_0?

$\Rightarrow$ Yes!

Thm (Mallat '89)

Let $\{V_j\}_{j\in\mathbb{Z}}$ be an MRA of $L^2(\mathbb{R})$.
Let $\phi(x)$ be the scaling fcn whose Fourier transf. is

$$\hat{\phi}(\xi) = \frac{\hat{\theta}(\xi)}{\left(\sum_{k\in\mathbb{Z}} |\hat{\theta}(\xi-k)|^2\right)^{1/2}}$$

division by this quantity is often called the orthogonalization trick.

Then, $\{\phi_{j,k}\}_{k\in\mathbb{Z}}$ form an ONB for V_j, $\forall j\in\mathbb{Z}$.

(Pf) Let $\phi \in V_0$.

$\Leftrightarrow \quad \phi(x) = \sum_{k\in\mathbb{Z}} a_k \theta(x-k), \quad \exists \{a_k\} \in \ell^2(\mathbb{Z})$

$\Leftrightarrow \quad \hat{\phi}(\xi) = \hat{a}(\xi)\,\hat{\theta}(\xi)$, $\quad \hat{a}$: 1-periodic Fourier series and $\hat{a} \in L^2[-\frac{1}{2}, \frac{1}{2}]$.

We want to impose the orthogonality on ϕ, i.e.,

$$\langle \phi(\cdot-k), \phi(\cdot-\ell) \rangle = \delta_{k\ell}$$

But
$$\langle \phi(\cdot-k), \phi(\cdot-\ell)\rangle = \int_{-\infty}^{\infty} \phi(x-k)\overline{\phi(x-\ell)}\,dx$$
$$= (\phi * \tilde{\phi})(\ell-k)$$

where $\tilde{\phi}(x) = \overline{\phi(-x)}$

Note: $\mathcal{F}[\phi * \tilde{\phi}](\xi) = |\hat{\phi}(\xi)|^2$

Thus, $\{\phi(x-k)\}_{k\in\mathbb{Z}}$: an ONB

$$\iff \phi * \tilde{\phi}(k) = \delta_{k0}$$

$$\iff \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi-k)|^2 \equiv 1 \quad \text{a.e. } \xi\in\mathbb{R} \qquad \text{—} (*)$$

Hence <u>if</u> we choose

$$\hat{a}(\xi) = \Big(\sum_{k\in\mathbb{Z}} |\hat{\theta}(\xi-k)|^2\Big)^{-1/2}$$

<u>then</u> $(*)$ is satisfied.

$$\Rightarrow \hat{\phi}(\xi) = \frac{\hat{\theta}(\xi)}{\left(\sum_{k\in\mathbb{Z}} |\hat{\theta}(\xi-k)|^2\right)^{1/2}}, \quad \text{and}$$

this is well defined since $\sqrt{A} \le \hat{a}(\xi) \le \sqrt{B}$ ///

Using these scaling fcn's, we can easily construct P_{V_j} as

$$P_{V_j} f = \sum_{k\in\mathbb{Z}} \underbrace{\langle f, \phi_{j,k}\rangle}_{=: f_k^j} \phi_{j,k}$$

$$f_k^j := \langle f, \phi_{j,k}\rangle = \int f(x) \frac{1}{\sqrt{2^j}} \overline{\phi\left(\frac{x-2^jk}{2^j}\right)}\, dx$$

$$= (f * \tilde{\phi}_j)(2^jk)$$

$$\tilde{\phi}_j(x) = \frac{1}{\sqrt{2^j}} \phi\left(\frac{-x}{2^j}\right) = \delta_{2^j}\tilde{\phi}$$

So, $f * \tilde{\phi}_j \xrightarrow{\mathcal{F}} \hat{f}\cdot \delta_{2^{-j}}\hat{\tilde{\phi}} = \hat{f}(\xi)\cdot\sqrt{2^j}\,\overline{\hat{\phi}(2^j\xi)}$

$\Rightarrow f * \tilde{\phi}_j(2^jk)$ means low-pass filtering f followed by sampling at the rate 2^j.

Example 1. Haar's scaling fcn

Example 2. Shannon's scaling fcn

In these two cases, $\theta = \phi$ since $\{\theta(x-k)\}_{k\in\mathbb{Z}}$ are already orthonormal. //

Example 3. Spline scaling fcn.

$$\hat{\phi}(\xi) = \frac{e^{-i\varepsilon\pi\xi}}{\xi^{m+1}\sqrt{S_{2m+2}(\xi)}}, \quad \varepsilon = \begin{cases} 0 & m = \text{odd} \\ 1 & m = \text{even} \end{cases}$$

where $S_n(\xi) := \sum_{k\in\mathbb{Z}} (\xi+k)^{-n}$

$\underline{m=0}$: $S_2(\xi) = \sum_{k\in\mathbb{Z}} (\xi+k)^{-2} = \pi^2 \operatorname{cosec}^2 \pi\xi = \left(\frac{\pi}{\sin\pi\xi}\right)^2$

$$\Rightarrow \hat{\phi}(\xi) = e^{-i\pi\xi}\frac{1}{\xi}\cdot\frac{\sin\pi\xi}{\pi} = e^{-i\pi\xi}\operatorname{sinc}(\xi)$$

$\underline{m=1}$: $S_4(\xi) = \sum (\xi+k)^{-4}$

how to get this?

$\Rightarrow S_2''(\xi) = 6\sum(\xi+k)^{-4}$, i.e.,

$$S_4(\xi) = \frac{1}{6}S_2''(\xi) = \cdots = \frac{\pi^4}{3}\,\frac{1+2\cos^2\pi\xi}{\sin^4\pi\xi}$$

$$\Rightarrow \hat{\phi}(\xi) = \frac{\sqrt{3}}{\sqrt{1+2\cos^2\pi\xi}}\left(\frac{\sin\pi\xi}{\pi\xi}\right)^2$$

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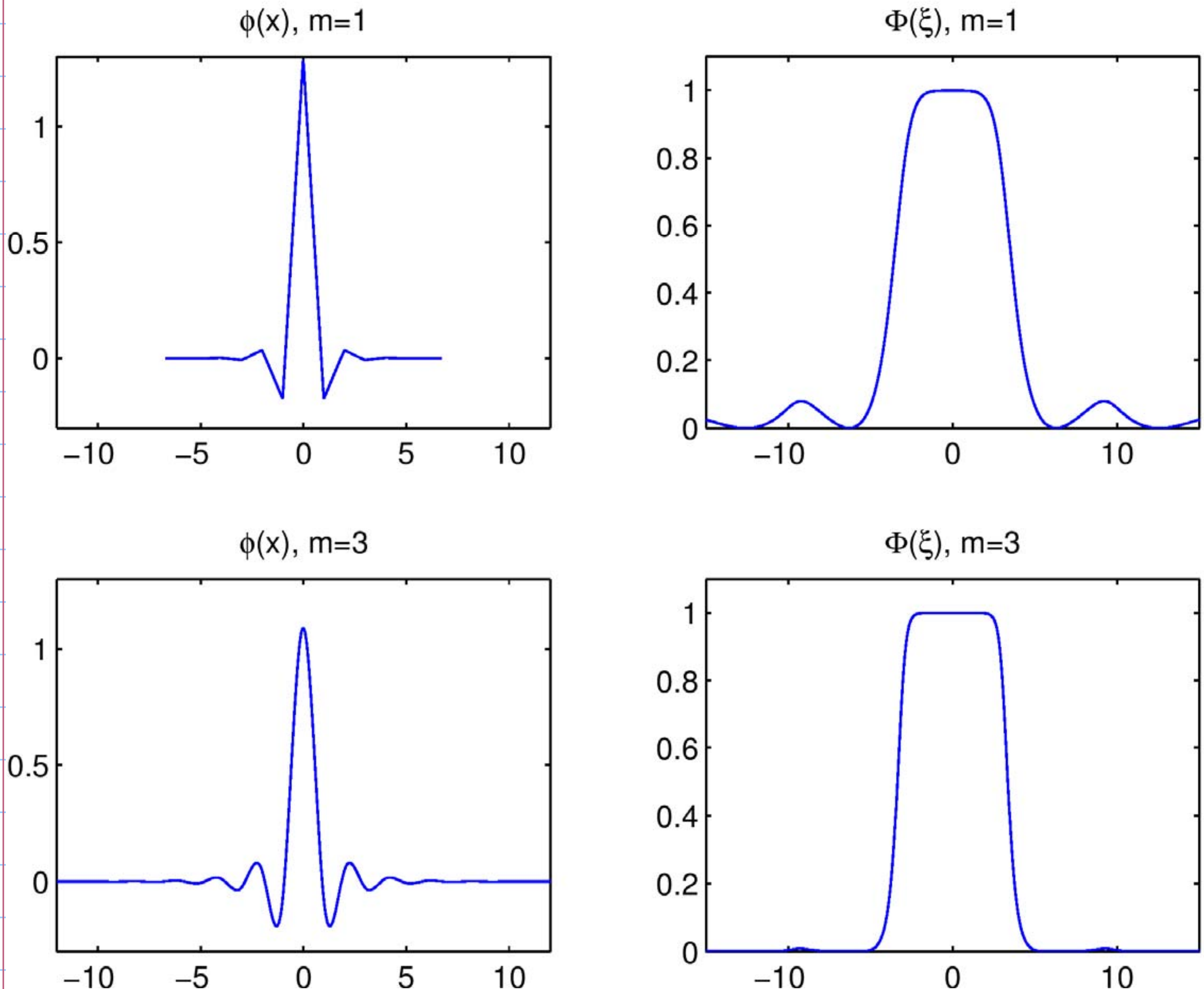
φ(x), m=1
Φ(ξ), m=1
φ(x), m=3
Φ(ξ), m=3
−10 −5 0 5 10
0 0.5 1
0 0.2 0.4 0.6 0.8 1
−10 0 10