Lecture 17: Wavelet Bases III

Note Title 3/2/2014

Recall

Thm (Mallat, Meyer 1986)
Let ϕ be a scaling fcn (father wavelet) and $\{h_k\}_{k\in\mathbb{Z}}$ be the corresponding CMF. Let us define $\psi \in L^2(\mathbb{R})$ whose Fourier transf. has

$$\hat{\psi}(\xi) = \frac{1}{\sqrt{2}}\,\hat{g}\left(\frac{\xi}{2}\right)\hat{\phi}\left(\frac{\xi}{2}\right)$$

with $\hat{g}(\xi) = e^{-2\pi i \xi}\,\overline{\hat{h}\left(\xi+\frac{1}{2}\right)}$.

Let $\psi_{j,k}(x) = 2^{-j/2}\psi(2^{-j}x-k)$.

Then, $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ form an ONB of W_j for each $j\in\mathbb{Z}$, and $\{\psi_{j,k}\}_{(j,k)\in\mathbb{Z}^2}$ form an ONB of $L^2(\mathbb{R})$.

Remarks:

(1) Let's check the relationship between $\{g_k\}_{k\in\mathbb{Z}}$ and $\{h_k\}_{k\in\mathbb{Z}}$.

But, first, recall

$$\begin{cases} h_k = \langle \phi_{1,0}, \phi_{0,k}\rangle = \langle \phi(\tfrac{\cdot}{2}), \phi(\cdot-k)\rangle \\ g_k = \langle \psi_{1,0}, \phi_{0,k}\rangle = \langle \psi(\tfrac{\cdot}{2}), \phi(\cdot-k)\rangle \end{cases}$$

from the two-scale diff. eqn's. and $V_1 \oplus W_1 = V_0$.

Now,

$$\hat{g}(\xi) = e^{-2\pi i\xi}\,\overline{\hat{h}\left(\xi+\frac{1}{2}\right)}$$

This choice of $\hat{g}$ satisfies all the cond's as $\hat{g}$.

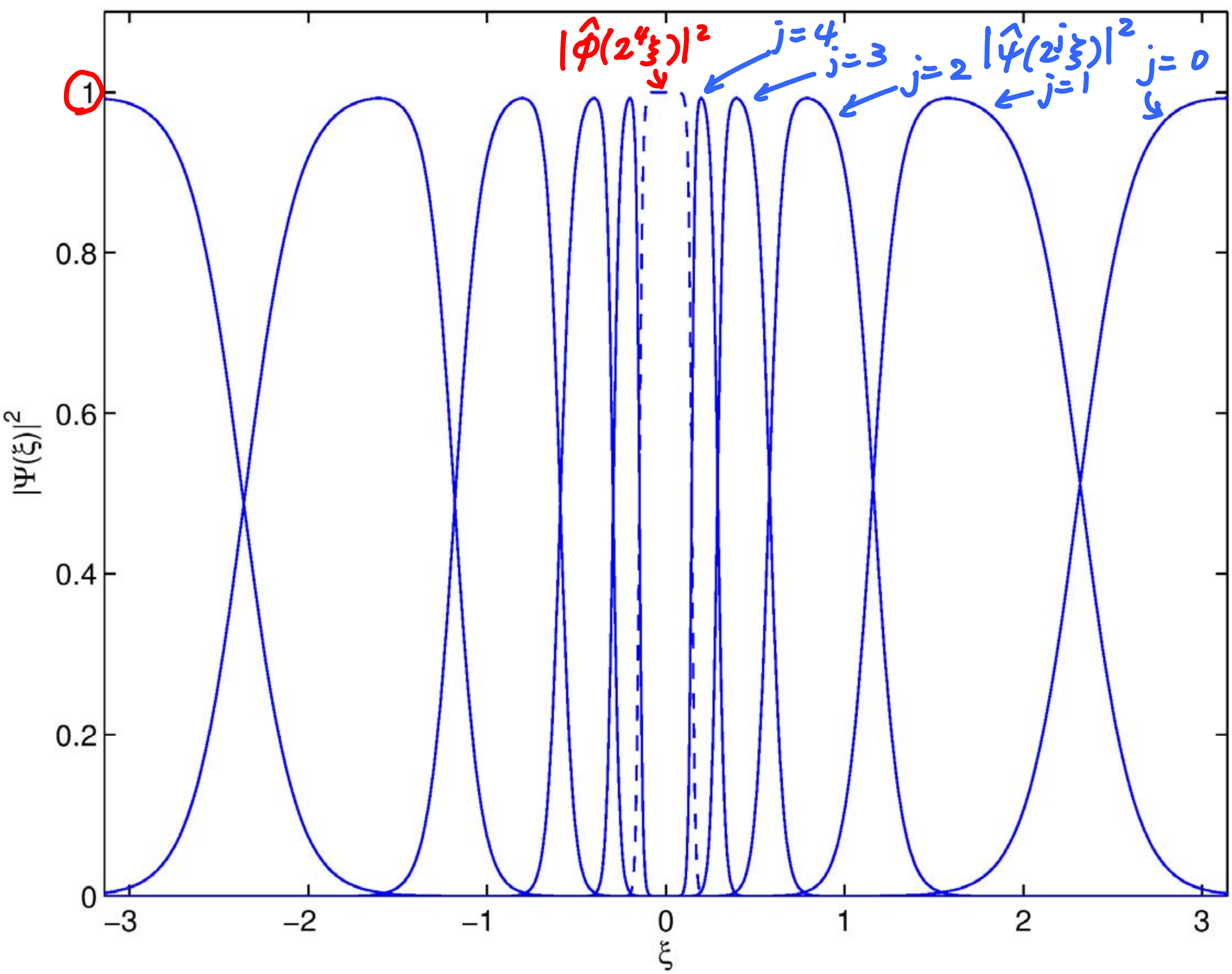
via $\mathcal{F}^{-1}$

$$g_k = (-1)^{1-k}\,h_{1-k}, \quad k\in\mathbb{Z}.$$

$$\hat{g}(\xi) = e^{-2\pi i \xi}\,\overline{\hat{h}(\xi+\tfrac{1}{2})}$$
$$= e^{-2\pi i \xi} \sum h_\ell\, e^{+2\pi i \ell(\xi+\frac{1}{2})}$$
$$= e^{-2\pi i \xi} \sum h_\ell\, e^{+2\pi i \ell \xi} \cdot e^{\pi i \ell} \quad = (-1)^\ell$$
$$= \sum (-1)^\ell h_\ell\, e^{2\pi i(\ell-1)\xi}$$
$1-\ell = k \longrightarrow$
$$= \sum_k \underbrace{(-1)^{1-k} h_{1-k}}_{= g_k} e^{-2\pi i k \xi}$$

(2) $|\hat{\psi}(2^j\xi)|^2$ works as a bandpass filter.

also $\displaystyle\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 \equiv 1$ a.e. $\xi \in \mathbb{R}\setminus\{0\}$



Example 1 : Haar wavelet

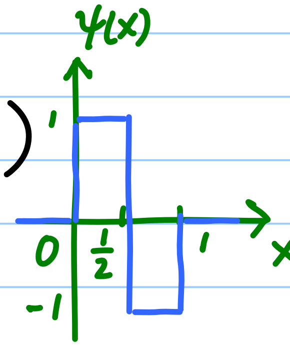
$$\phi(x) = \chi_{[0,1)}(x) \Rightarrow \psi(x) = \begin{cases} -1 & 0<x<\frac{1}{2} \\ 1 & \frac{1}{2}<x<1 \\ 0 & \text{o.w.} \end{cases}$$

Recall $h_k = \begin{cases} 1/\sqrt{2} & k=0,1 \\ 0 & \text{o.w.} \end{cases}$

$$\Rightarrow \frac{1}{\sqrt{2}}\psi\left(\frac{x}{2}\right) = \sum (-1)^{1-k} h_{1-k}\,\phi(x-k)$$

$$= -h_1\,\phi(x) + h_0\,\phi(x-1)$$

$$= \frac{1}{\sqrt{2}}\left(-\chi_{[0,1)}(x) + \chi_{[1,2)}(x)\right)$$



$$\Rightarrow \psi(x) = -\chi_{[0,\frac{1}{2})}(x) + \chi_{[\frac{1}{2},1)}(x)$$

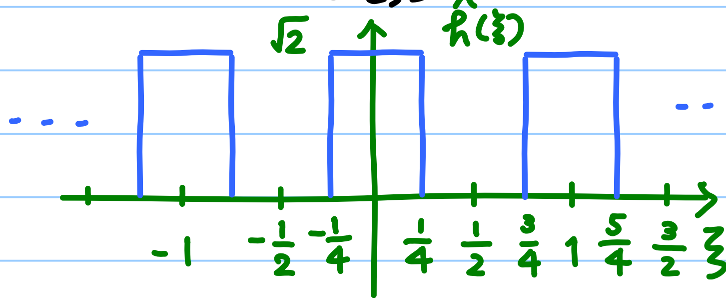
$$= \frac{1}{\sqrt{2}}\left[\sqrt{2}\,\phi(2x-1) - \sqrt{2}\,\phi(2x)\right].$$

Example 2 : Shannon wavelet

Recall : $\hat{\phi}(\xi) = \chi_{[-\frac{1}{2},\frac{1}{2})}(\xi)$, $\hat{h}(\xi) = \sqrt{2}\,\chi_{[-\frac{1}{4},\frac{1}{4})}(\xi)$

1 - periodic

$$= \sqrt{2}\sum_k \chi_{[-\frac{1}{4},\frac{1}{4})}(\xi+k)$$



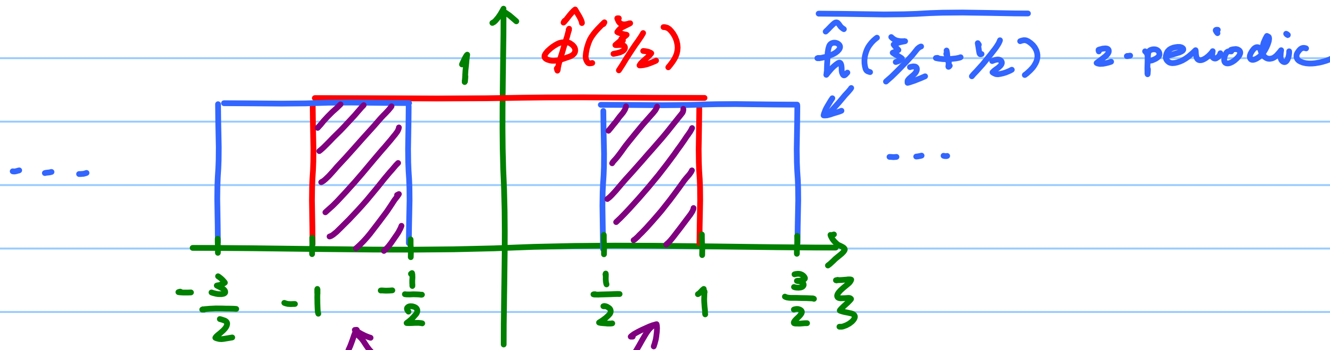
$$\hat{\psi}(\xi) = \frac{1}{\sqrt{2}}\,\hat{g}\left(\frac{\xi}{2}\right)\hat{\phi}\left(\frac{\xi}{2}\right) = \frac{1}{\sqrt{2}}\,e^{-i\pi\xi}\,\overline{\hat{h}\left(\frac{\xi}{2}+\frac{1}{2}\right)}\,\hat{\phi}\left(\frac{\xi}{2}\right)$$

$$= e^{-i\pi\xi}\,\underbrace{\chi_{[-\frac{3}{2},-\frac{1}{2})}(\xi)}_{\text{2 - periodic}}\,\chi_{[-1,1)}(\xi)$$

" $\sqrt{2}\,\chi_{[-\frac{3}{2},-\frac{1}{2})}(\xi)$

2 - periodic

$$= e^{-i\pi\xi}\left(\chi_{[-1,1)}(\xi) - \chi_{[-\frac{1}{2},\frac{1}{2})}(\xi)\right)$$



$|\hat{\psi}(\xi)|$ or $e^{i\pi\xi}\hat{\psi}(\xi)$
ideal (or pure) bandpass filter (could be too sharp in practice).

Hence via $\mathcal{F}^{-1}$, we have

$$\psi(x) = 2\,\mathrm{sinc}(2x-1) - \mathrm{sinc}(x-\tfrac{1}{2})$$

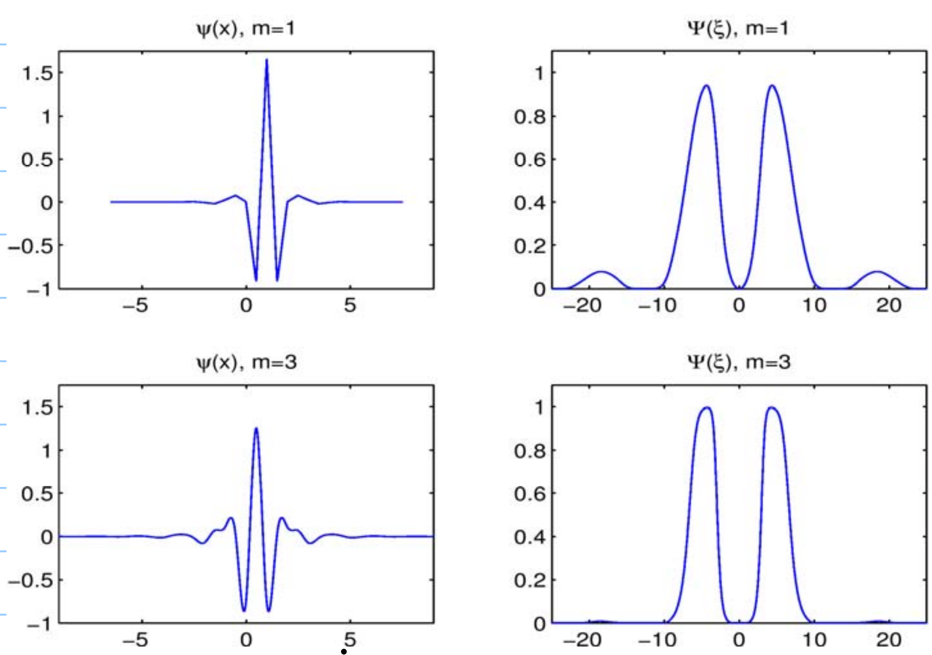
$$= \sqrt{2}\left[\sqrt{2}\,\phi(2x-1) - \frac{1}{\sqrt{2}}\,\tau_{\frac{1}{2}}\phi(x)\right]$$

Example 3: Spline (a.k.a. Battle-Lemarié) wavelet

$m=0$: the Haar wavelet

For general $m \in \mathbb{N}$

$$\hat{\psi}(\xi) = \frac{e^{-i\pi\xi}}{\xi^{m+1}}\sqrt{\frac{S_{2m+2}(\xi/2+1/2)}{S_{2m+2}(\xi)\,S_{2m+2}(\xi/2)}}$$



☆ Basic Properties of Wavelets

⇒ Important in choosing/designing wavelets.

(1) Vanishing Moments

Def. A mother wavelet ψ is said to have p vanishing moments if

$$\int_{-\infty}^{\infty} x^m \psi(x)\,dx = 0, \quad m = 0, 1, \cdots, p-1$$

i.e., $\psi \perp \mathcal{P}_{p-1}[x]$ (any polynomial with deg. $\leq p-1$).

If your fcn $f(x)$ is locally C^m, then f is well-approximated by a Taylor polynomial of deg m over some short interval.

⇒ $\langle f, \psi_{j,k} \rangle \approx 0$ for small j if $m < p$.

So, in general, the larger p, the better.

Thm (Vanishing moments)

Let ϕ & ψ be father & mother wavelets generating a wavelet ONB for $L^2(\mathbb{R})$ with $|\phi(x)| = O((1+x^2)^{-p/2-1})$, $|\psi(x)| = O((1+x^2)^{-p/2-1})$.

Then the following statements are equivalent:

(a) ψ has p vanishing moments.

(b) $\hat{\psi}^{(m)}(0) = 0, \quad m = 0, 1, \cdots, p-1$.

(c) $\hat{h}^{(m)}(\frac{1}{2}) = 0, \quad m = 0, 1, \cdots, p-1$.

(d) $\forall\ 0 \leq m < p$,

$$g_m(x) := \sum_{k \in \mathbb{Z}} k^m \phi(x-k)$$ is a polynomial of deg. m.

(Proof) See Mallat's book Sec. 7.2. for the detail.

Note that

$$\hat{\psi}^{(m)}(\xi) = \mathcal{F}[(-2\pi i x)^m \psi(x)] \Rightarrow \hat{\psi}^{(m)}(0) = (-2\pi i)^m \int_{-\infty}^{\infty} x^m \psi(x)\,dx$$

(2) Support Size

If your f has an isolated singularity at $x = x_0$ and if $x_0 \in \operatorname{supp} \psi_{j,k}$, then $|\langle f, \psi_{j,k} \rangle|$ may be large.

$\Rightarrow$ better to have $\operatorname{supp}\psi$: compact !

Thm (Compact Support)

The father wavelet ϕ has a compact support $\Longleftrightarrow$ $\{h_k\}$ has a compact support (i.e., finite taps). Their supports are equal.

a slight abuse of def. of supp.

If $\underbrace{\operatorname{supp}\phi}_{\text{interval}} \overset{\star}{=} \underbrace{\operatorname{supp} h}_{\text{list of indices i.e., } \{k \in \mathbb{Z} \mid h_k \neq 0\}} = [K_1, K_2]$, $K_1, K_2 \in \mathbb{Z}$,

then $\operatorname{supp}\psi = [(K_1 - K_2 + 1)/2, (K_2 - K_1 + 1)/2]$.

(Proof) ($\Rightarrow$) Recall $h_k = \langle \phi_{1,0}, \phi_{0,k} \rangle$.

Since both $\phi_{1,0}$ & $\phi_{0,0}$ have compact supports, $\{h_k\}$ must be of finite taps.

($\Leftarrow$) See Daubechies (1988, Comm. Pure & Appl. Math.)

Now, check the supports of ϕ and $\{h_k\}$.

Suppose $h_k \neq 0$ for $K_1 \le k \le K_2$, and $\operatorname{supp}\phi = [N_1, N_2]$. Then, $\operatorname{supp}\phi_{1,0} = [2N_1, 2N_2]$

On the other hand,

$$\phi_{1,0}(x) = \frac{1}{\sqrt{2}}\phi\left(\frac{x}{2}\right) = \sum_{k\in\mathbb{Z}} h_k \phi(x-k)$$

$\Rightarrow$ $\operatorname{supp}\phi_{1,0} = [K_1 + N_1, K_2 + N_2]$

Hence $2N_1 = K_1 + N_1$, $2N_2 = K_2 + N_2$

i.e., $N_1 = K_1$, $N_2 = K_2$. ✓

As for $\operatorname{supp}\psi$, recall

$$\psi_{1,0}(x) = \frac{1}{\sqrt{2}}\psi\left(\frac{x}{2}\right) = \sum_{k\in\mathbb{Z}} g_k \phi(x-k) = \sum_{k\in\mathbb{Z}} (-1)^{1-k} h_{1-k}\phi(x-k)$$

Since $\operatorname{supp}\phi = \operatorname{supp} h = [K_1, K_2]$, the support of the above two-scale diff. eqn. is $[K_1 - K_2 + 1, K_2 - K_1 + 1]$, which is $\operatorname{supp} \psi_{1,0}$.

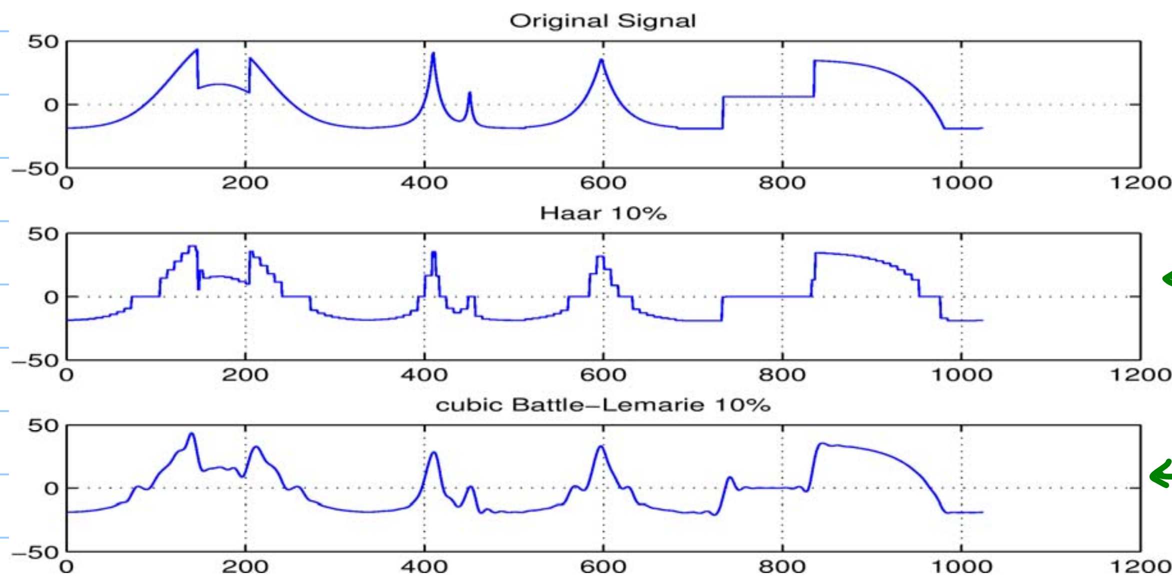
$1-k \in [K_1, K_2] \iff k \in [1-K_2, 1-K_1]$

Hence, $\operatorname{supp}\psi = [(K_1 - K_2 + 1)/2, (K_2 - K_1 + 1)/2]$ ///

<u>Remark</u>: The support size & the number of vanishing moments of ψ seems independent. However, Daubechies showed that for orthonormal wavelets, ψ has p vanishing moments $\Rightarrow |\operatorname{supp}\psi| \geq 2p-1$.
$\exists$ a trade-off between these two concepts. Daubechies's orthonormal wavelets achieve $|\operatorname{supp}\psi| = 2p-1$ with p vanishing moments.

(3) Regularity

When you reconstruct (or approximate) f from its compressed representation (e.g., from a finite subset of $\{\langle f, \psi_{j,k}\rangle\}$, the regularity (i.e., smoothness) of $\psi_{j,k}$ matters.



basis fcns are visible

Thm (Tchamitchian 1987)

Let $\hat{h}(\xi)$ be a CMF with p zeros at $\xi = \frac{1}{2}$ which satisfies the sufficient cond.'s of the Mallat - Meyer Thm, i.e.,

1) 1 - periodic;
2) C^1 in the neighborhood of $\xi = 0$; and
3) $|\hat{h}(\xi)|^2 + |\hat{h}(\xi+\frac{1}{2})|^2 \equiv 2$; $\hat{h}(0) = \sqrt{2}$;
$\inf_{\xi \in [-\frac{1}{4}, \frac{1}{4}]} |\hat{h}(\xi)| > 0$.

Now, let's perform the factorization:

$$\hat{h}(\xi) = \sqrt{2} \left(\frac{1 + e^{i 2\pi \xi}}{2} \right)^p \hat{\ell}(\xi)$$

If $\sup_{\xi \in \mathbb{R}} |\hat{\ell}(\xi)| = B$, then ϕ & ψ are uniformly Lipschitz (or Hölder) α for

$$\alpha < \alpha_0 := p - \log_2 B - 1 .$$

(Proof) See Mallat's book Sec. 7.2.

Note that we refine the definition of Lipschitz continuity in Lecture 5 as follows:

Def A fcn f is pointwise Lipschitz $\alpha \geq 0$ at x_0 if $\exists K > 0$, $\exists p_{x_0} \in \mathcal{P}_{\lfloor \alpha \rfloor}[x]$ s.t.

(*) $|f(x) - p_{x_0}(x)| \leq K |x - x_0|^\alpha$, $\forall x \in \mathbb{R}$.

A fcn f is uniformly Lipschitz α over $[a, b]$ if it satisfies (*) for $\forall x_0 \in [a, b]$ and K is independent of x_0.

The space of uniformly Lipschitz continuous fcns with exponent α is often written as $\mathrm{Lip}_\alpha[a, b]$. But if $\alpha = m + \beta$, $m \in \mathbb{N} \cup \{0\}$, $0 < \beta < 1$

it is also written as $C^{m,\beta}[a,b]$, i.e.,

$f \in C^{m,\beta}[a,b] \Leftrightarrow f \in C^{m}[a,b]$ & $f^{(m)} \in C^{0,\beta}[a,b] = Lip_\beta[a,b]$.

Ex. $f(x) = x^\gamma \in C[0,1]$, $0 < \gamma \leq 1$

$\notin Lip_\alpha[0,1]$ with $\gamma < \alpha \leq 1$, but

$\in Lip_\alpha[0,1]$ with $0 < \alpha \leq \gamma$

Let's check these properties in Examples 1, 2, 3.

Example 1 : Haar Wavelets

- Vanishing moments $p = 1$.
- $\text{Supp}\, \psi = [0,1)$
- Regularity : discontinuous
- Antisymmetric

Example 2 : Shannon Wavelets

- Vanishing moments $p = +\infty$ (because $\hat{\psi}^{(m)}(0) \equiv 0$ $\forall m \in \mathbb{N}$)
- $\text{Supp}\, \psi = \mathbb{R}$, slow decay $\sim \frac{1}{x}$
- Regularity : $C^\infty(\mathbb{R})$
- Symmetric

Example 3 : Battle-Lemarié Wavelets

- Vanishing moments $p = m+1$
- $\text{Supp}\, \psi = \mathbb{R}$, exponential decay
- Regularity : $C^{m-1}(\mathbb{R})$
- Symmetric

☆ A Brief Intro. to Compactly Supported Wavelets of Daubechies

Daubechies's wavelets have the following properties:

- Vanishing moments $p \in \mathbb{N}$.
- $|\mathrm{Supp}\,\psi| = 2p-1$ minimal supp for a given p.
- Regularity : $\mathrm{Lip}_\alpha(\mathbb{R})$, $\alpha < p - \log_2 B - 1$ $\approx p/5$ for large p.
- Neither symmetric nor antisymmetric if $p > 1$ ($p=1$ reduces to the Haar case).

The name of the game here is how to find (or design) a trigonometric polynomial $R(e^{-i2\pi\xi})$ of minimum degree s.t.

$$\hat{h}(\xi) = \sqrt{2}\left(\frac{1+e^{-i2\pi\xi}}{2}\right)^p R(e^{-2\pi i\xi})$$

with $|\hat{h}(\xi)|^2 + |\hat{h}(\xi+\tfrac{1}{2})|^2 \equiv 2$ a.e. $\xi \in \mathbb{R}$

$$\hat{h}(0) = \sqrt{2}.$$

Daubechies proved the minimum deg $= p-1$.

Thm (Daubechies 1988)

A real-valued CMF $\{h_k\}$ with $\hat{h}(\xi)$ having p zeros at $\xi = 1/2$ has at least $2p$ nonzero coefficients (i.e., $|\mathrm{supp}\,h| \geq 2p$, $|\mathrm{supp}\,\psi| \geq 2p-1$).

Daubechies's wavelet filters have $2p$ taps.

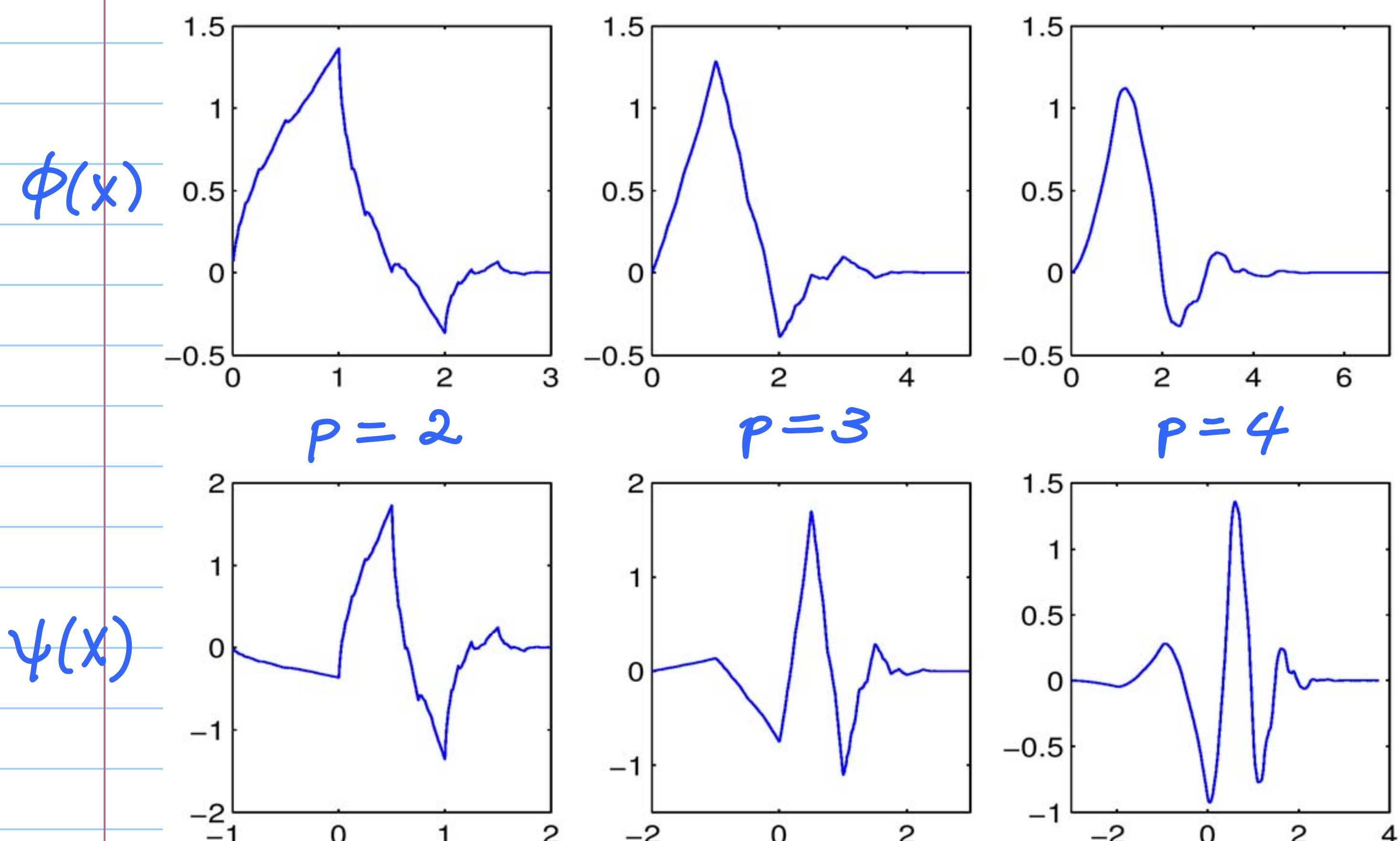
(Proof) See Daubechies 1988 or Mallat's book Sec. 7.2.

Thm (Daubechies 1988)

<u>If</u> ψ is a mother wavelet with p vanishing moments that generates an ONB of $L^2(\mathbb{R})$, <u>then</u> $|\operatorname{supp}\psi| \geq 2p-1$.

A Daubechies mother wavelet has a minimum size support $\operatorname{supp}\psi = [-p+1, p]$ and the corresponding father wavelet has $\operatorname{supp}\phi = [0, 2p-1]$.

(Proof) See Daubechies 1988 or Mallat's book Sec. 7.2.



$$\left.\begin{aligned} h_0 &= 0.482962913144831 \\ h_1 &= 0.836516303737708 \\ h_2 &= 0.224143868041922 \\ h_3 &= -0.129409522550955 \end{aligned}\right\} \quad p = 2 \text{ case.}$$