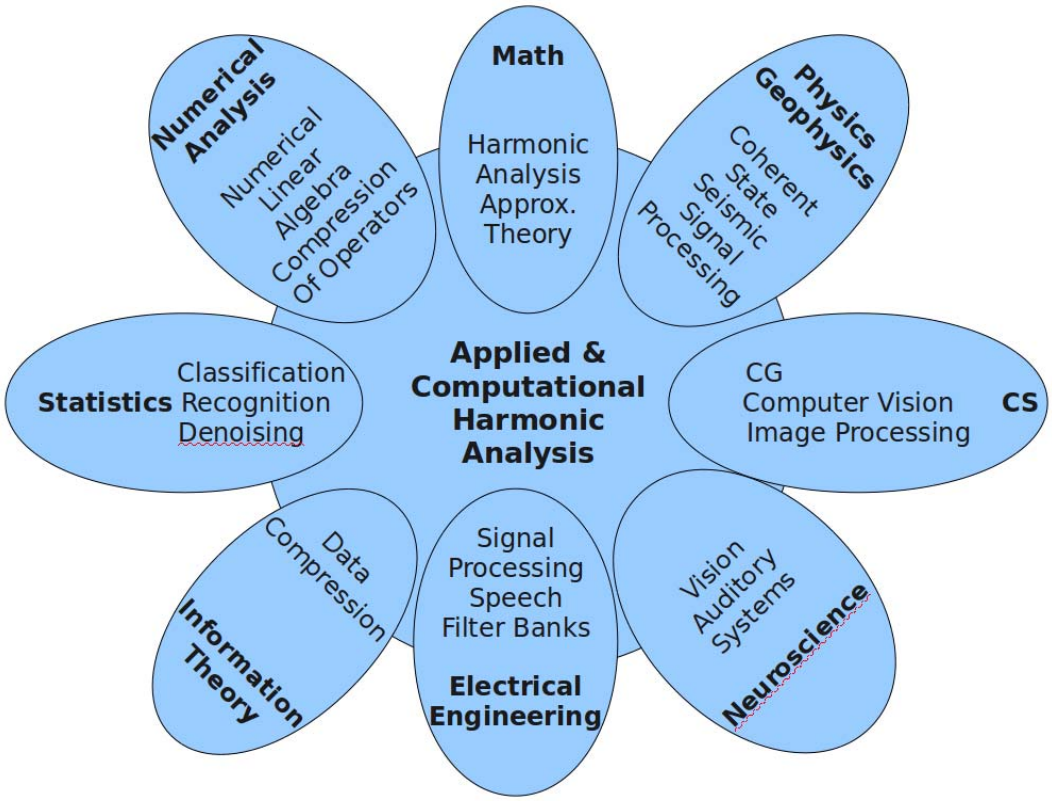
MAT 271: Applied & Computational Harmonic Analysis

Note Title

Lecture 1 : Overture

Applied & Computational Harmonic Analysis (ACHA) is an extremely interdisciplinary field!



This is not surprising considering the ubiquity of the celebrated Fourier transform!

The key role of ACHA

→ provides great tools for representing your data/functions/operators.

* Representation of data/functions using ACHA-based tools is:

(1) Efficient → good for compression/approximation

$$f(x) \approx \sum_{k=1}^{n} \alpha_k \varphi_k(x)$$

where n: small, $\{\varphi_k\}$: an ONB

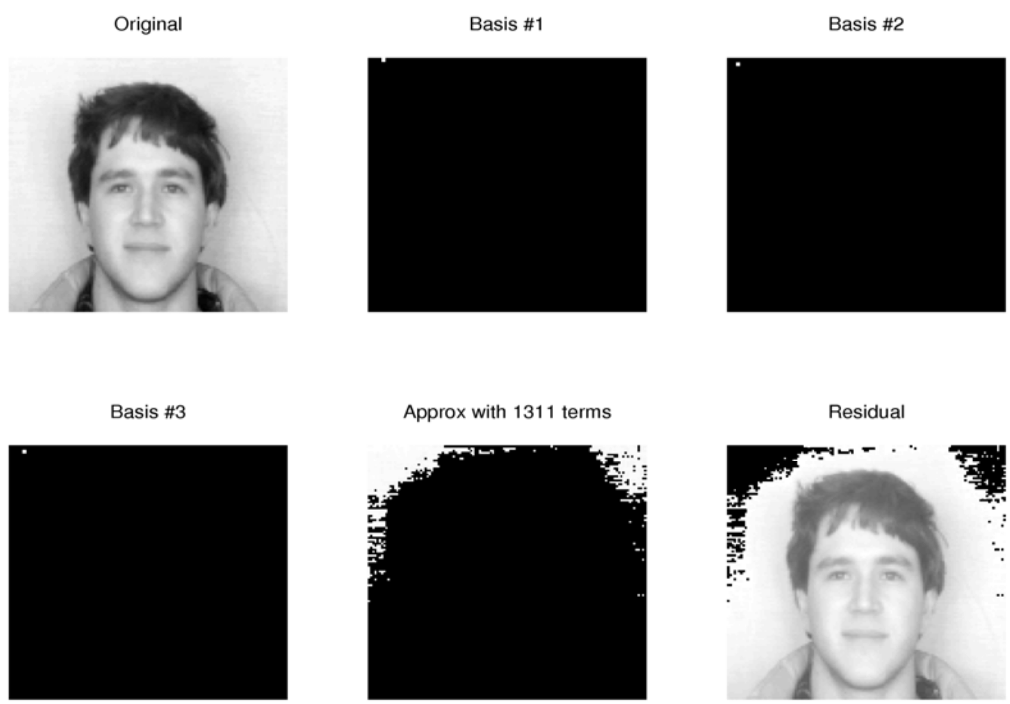
or more precisely,

$$\left\| f - \sum_{k=1}^{n} \alpha_k \varphi_k \right\| = O(n^{-\beta})$$

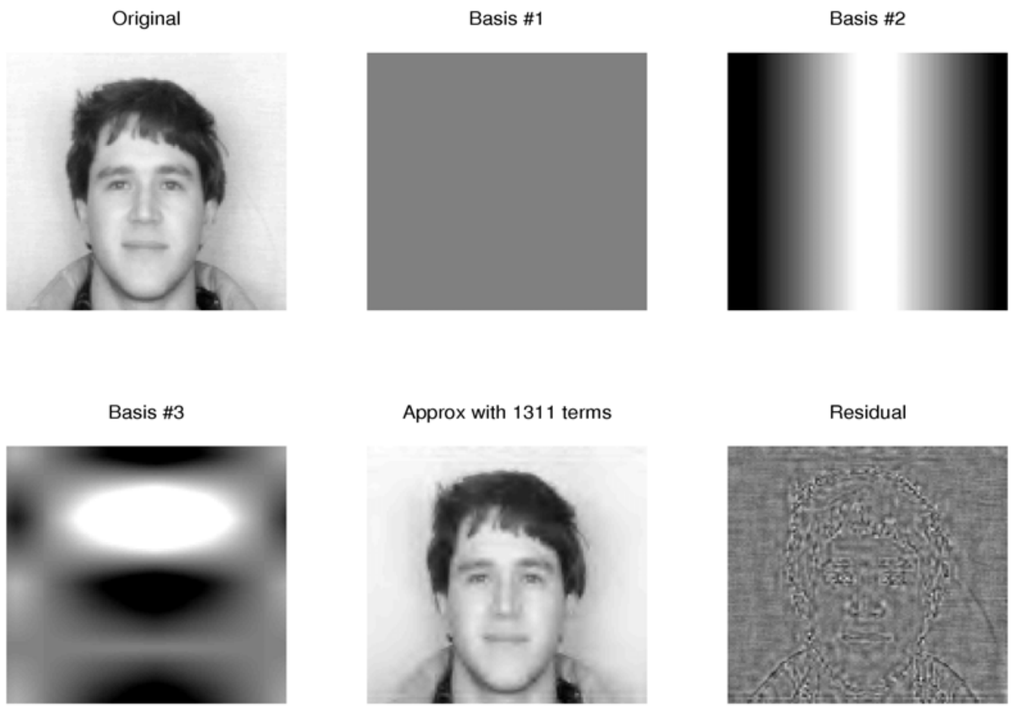
$$\exists \beta > 0$$

what classes of fcns possess such rate of approximation for a given set of $\{\varphi_k\}$?

A bad example:

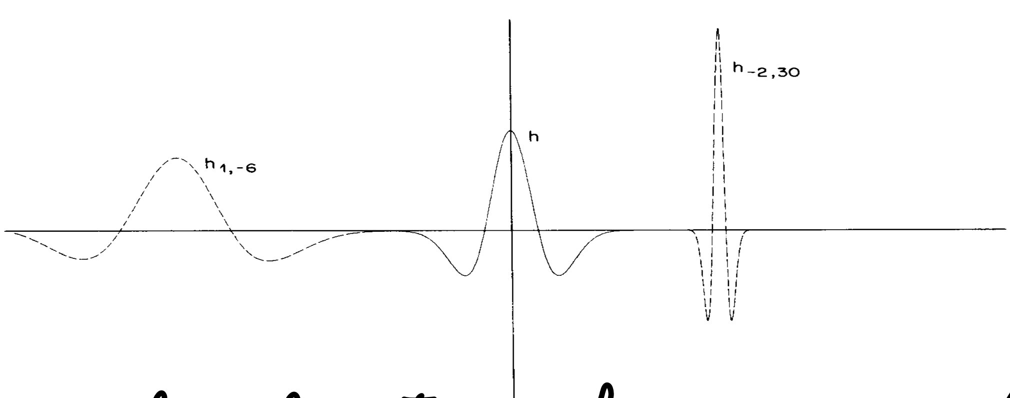


A good example:



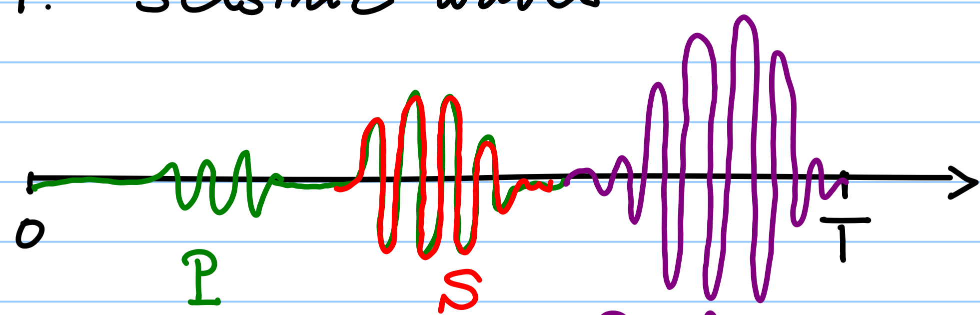
(2) Meaningful / useful

- $\{\alpha_k\}$ are used for interpretation, classification, discrimination, ...
- $\{\varphi_k\}$ are easily interpretable.

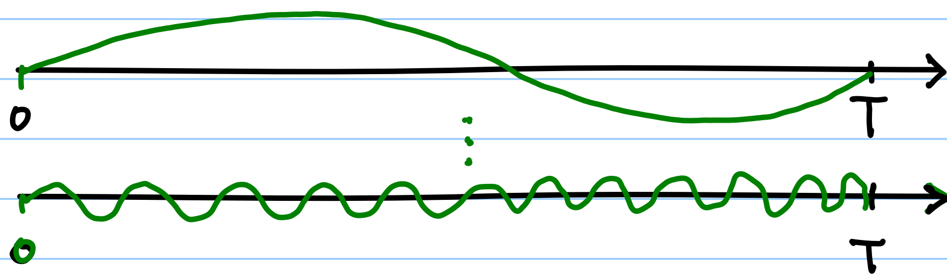


specific location, frequency, scale, duration, ...

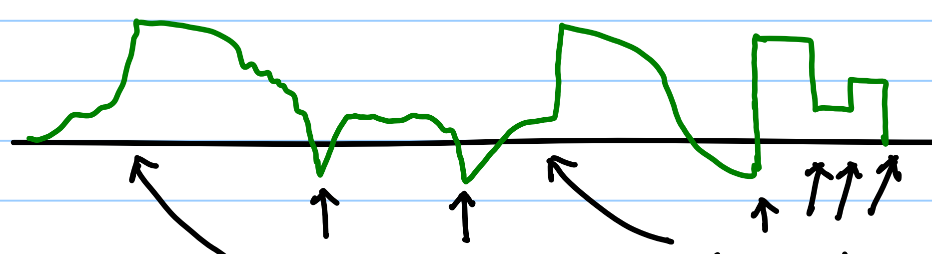
Ex. 1. Seismic waves



→ Wrong to interpret such a signal with global sinusoids



Ex. 2 images (profiles of images)



cusps   sharp discontinuities

(discont. in derivatives)

→ Wrong to use <u>global sinusoids</u>

→ less efficient

coeff's do not decay fast ...

(3) Representation should be adapted to your task!

Representation (or feature extraction) for compression may be different from that for classification
$\Rightarrow$ potraits of twins

(4) Computationally fast & stable

- If $\{\varphi_k\}$ are orthonormal, then
  $$\alpha_k = \langle f, \varphi_k \rangle = \int f(x) \overline{\varphi_k(x)}\, dx$$

$\Rightarrow$ Matrix - vector multiplication in general.

$$\boldsymbol{\alpha} = \Phi^* \boldsymbol{f}$$

which requires $O(n^2)$ op's : slow!
Want to have $O(n \log n)$ or faster!

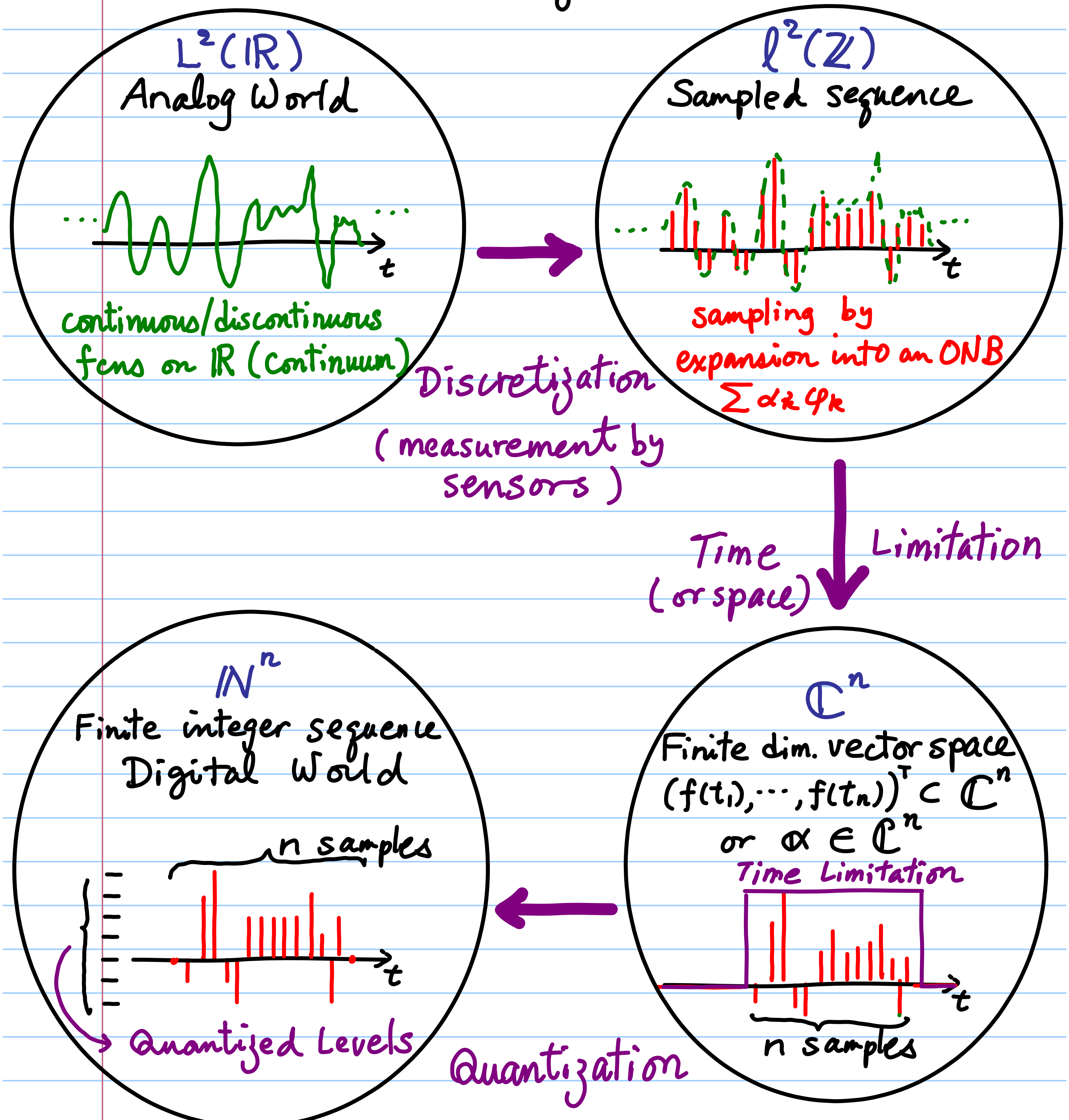
(5) Representation allows us to follow a very important strategy for any data / signal / image analysis tasks

It's a philosophy called

Ulf Grenander "Analysis by Synthesis"
Pattern Theory

We can truly "analyze" or "understand" the data / signal / images by looking at how they are composed of elementary "molecules" or "atoms".

* What is a signal?



* We won't discuss the quantization procedure in this course.
* Sampling can be viewed as an expansion of a signal into a special ONB $\Rightarrow$ The Shannon Sampling Thm

* Until we discuss Harmonic Analysis on graphs & networks, we assume that our signals are supported on

$\mathbb{R}^d$, $d \geq 1$ (unbdd, continuum);

$\prod_{j=1}^{d} (a_j, b_j) \subset \mathbb{R}^d$ (a rectangle);

$\mathbb{Z}^d$, $d \geq 1$ (unbdd, discrete); or

$\prod_{j=1}^{d} \mathbb{Z}_{n_j}$, $n_j \in \mathbb{N}$ (regular lattice)

$(0, 1, \cdots, n_1 - 1) \times (0, 1, \cdots, n_2 - 1) \times \cdots \times (0, 1, \cdots, n_d - 1)$