

Lecture 2: Basics of Fourier Transforms

Note Title

We will mainly focus on 1D signals.

1. Fourier Transform on $L^1(\mathbb{R})$

Let $f \in L^1(\mathbb{R}) := \{f: \mathbb{R} \rightarrow \mathbb{C} \mid \|f\|_1 < \infty\}$

where $\|f\|_1 := \int_{-\infty}^{\infty} |f(x)| dx$ is called

the L^1 -norm of f . This integral is that of Lebesgue

Def. The **Fourier transform** of $f \in L^1(\mathbb{R})$ is the function \hat{f} on \mathbb{R} defined by

$$(*) \quad \hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx$$

We also write $\mathcal{F}[f](\xi) = \hat{f}(\xi)$

(Note that in \mathbb{R}^d , the definition becomes
$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle \xi, x \rangle} dx$$
 where $\langle \cdot, \cdot \rangle$ is the standard inner product in \mathbb{R}^d .)

Now, (*) is well defined, i.e., abs. conv.:

$$|\hat{f}(\xi)| \leq \int_{-\infty}^{\infty} |f(x)| dx = \|f\|_1 < \infty$$

since $|e^{-2\pi i \xi x}| = 1$. //

In fact, $\mathcal{F}: L^1(\mathbb{R}) \rightarrow BC(\mathbb{R}) = C(\mathbb{R}) \cap L^\infty(\mathbb{R})$

* Basic Properties of the Fourier transform

Def. For any $a \in \mathbb{R}$, the **translation operator** T_a is defined by

$$T_a f(x) := f(x - a)$$

Def. For any $s > 0$, the **dilation operator** \mathcal{D}_s is defined by

$$\mathcal{D}_s f(x) = \frac{1}{\sqrt{s}} f\left(\frac{x}{s}\right)$$

\searrow This factor guarantees that \mathcal{D}_s is an **isometry** in $L^2(\mathbb{R})$
i.e., $\|f\|_2 = \|\mathcal{D}_s f\|_2$

We'll do more of L^2 shortly.

$$\|f\|_2 := \sqrt{\int_{-\infty}^{\infty} |f(x)|^2 dx}$$

Def. The **convolution** of $f, g \in L^1$, is defined as

$$f * g(x) := \int f(x-y)g(y) dy$$

Note: $f * g = g * f$

$(f * g) * h = f * (g * h)$, etc.

Thm 1. Let $f, g \in L^1$. Then,

$$(a) \mathcal{F}[\tau_a f] = e^{-2\pi i \xi a} \hat{f}(\xi)$$

$$\mathcal{F}[e^{2\pi i a x} f(x)] = \tau_a \hat{f}(\xi) = \hat{f}(\xi - a)$$

$$(b) \mathcal{F}[\delta_s f] = \sqrt{s} \hat{f}(s\xi) = \delta_{1/s} \hat{f}(\xi)$$

clearly, $\mathcal{F}[\delta_{1/s} f] = \delta_s \hat{f}(\xi)$

$$(c) \mathcal{F}[f * g] = \hat{f} \hat{g}$$

$$\mathcal{F}[f g] = \hat{f} * \hat{g}$$

$$(d) \text{ Let } \partial_x := \frac{\partial}{\partial x}$$

If $f \in C^k$ and $\partial_x^j f \in L^1$, $j=1, \dots, k$
and $\partial_x^j f \in C_0$, $j=1, \dots, k-1$,

then

$$\mathcal{F}[\partial_x^k f] = (2\pi i \xi)^k \hat{f}(\xi)$$

\hookrightarrow continuous & vanishes at $\pm\infty$
 \Rightarrow used in

(e) On the other hand,

if $x^j f \in L^1$, $j=1, \dots, k$,

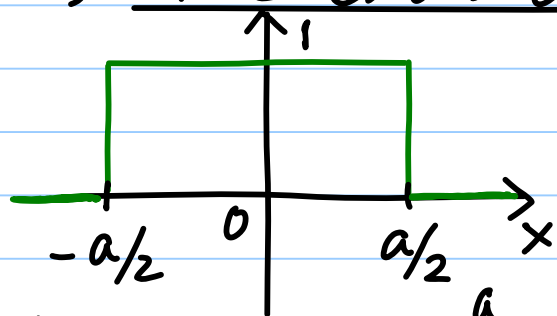
then $\hat{f} \in C^k$ and $\mathcal{F}[x^k f(x)] = \left(\frac{i}{2\pi}\right)^k \partial_\xi^k \hat{f}(\xi)$

integration by parts!

//

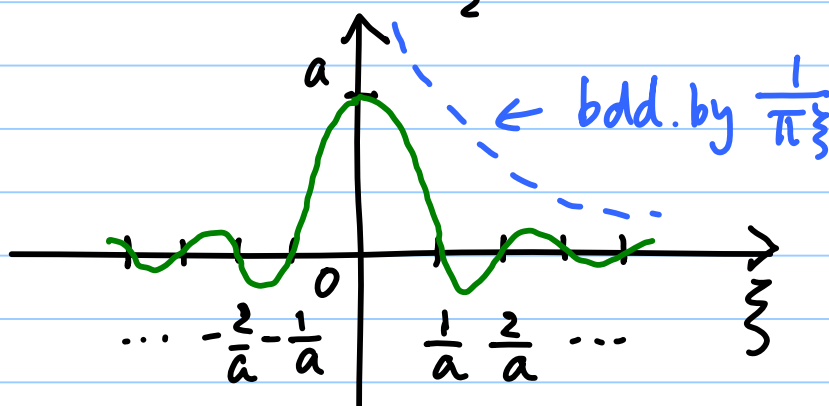
* Important Examples

(1) The characteristic (indicator) fun



$$\chi_{(-\frac{a}{2}, \frac{a}{2})}(x) := \begin{cases} 1 & \text{if } |x| < \frac{a}{2} \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{aligned} \hat{\chi}_{(-\frac{a}{2}, \frac{a}{2})}(\xi) &= \int_{-\frac{a}{2}}^{\frac{a}{2}} 1 \cdot e^{-2\pi i \xi x} dx = \frac{e^{-\pi i a \xi} - e^{\pi i a \xi}}{-2\pi i \xi} \\ &= \frac{\sin \pi a \xi}{\pi \xi} \end{aligned}$$



zeros are at
 $\xi = n/a, n \in \mathbb{Z} - \{0\}$.

If we define $\text{sinc}(x) := \frac{\sin \pi x}{\pi x}$,

then we can write

$$\hat{\chi}_{(-\frac{a}{2}, \frac{a}{2})}(\xi) = a \text{sinc}(a\xi)$$

Note that this fun is not in L^1 but in L^2 !

$$\odot \int_{-\infty}^{\infty} \left| \frac{\sin \pi a \xi}{\pi \xi} \right| d\xi = +\infty \text{ but } \int_{-\infty}^{\infty} \left| \frac{\sin \pi a \xi}{\pi \xi} \right|^2 d\xi = a < \infty$$

The details are left as an exercise. //

(2) The Gaussian fcn

$$g(x; \sigma) = g_\sigma(x) := \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}, \quad \sigma > 0$$

$x \in \mathbb{R}$

How to compute $\hat{g}_\sigma(\xi)$?

\Rightarrow HW problem! \exists a neat way.

$$\hat{g}_\sigma(\xi) = e^{-2\pi^2\sigma^2\xi^2}$$

Incidentally, in general, we have

$$\hat{f}(0) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i 0 \cdot x} dx = \int_{-\infty}^{\infty} f(x) dx$$

= The DC component of $f(x)$
 \uparrow EE terminology

Hence, $\hat{g}_\sigma(0) = 1$, i.e., $g_\sigma(x)$ is
a probability density fcn (pdf)

Also note that if $\sigma = \frac{1}{\sqrt{2\pi}}$, then

$$e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi \xi^2}$$

invariant w.r.t. \mathcal{F} !

* The Riemann-Lebesgue Lemma

If $f \in L^1$, then $\hat{f}(\xi) \rightarrow 0$ as $\xi \rightarrow \pm\infty$

(i.e., $\mathcal{F}[L^1(\mathbb{R})] \subset C_0(\mathbb{R})$)

"If $f \in L^1$, very high frequency components vanish." (non rigorous statement!)

* The Fourier Inversion Thm

states a procedure to recover f from \hat{f} .

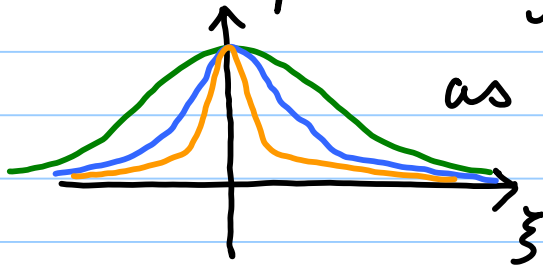
Want to have

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi$$

But the problem is:

\hat{f} may not be in L^1 . e.g., $\text{sinc} \notin L^1$

Hence, we use the "cut off" fcn
a good example is $g_a(\xi) = e^{-2\pi^2 a^2 \xi^2}$



as $a \uparrow$, more localized

Consider

$$(*) \int \underbrace{\hat{f}(\xi)} \hat{g}_\alpha(\xi) e^{2\pi i \xi x} d\xi$$
$$= \iint \underbrace{f(y) e^{-2\pi i \xi y}} dy e^{-2\pi^2 \sigma^2 \xi^2} e^{2\pi i \xi x} d\xi$$

OK
by
Fubini

$$= \iint f(y) e^{2\pi i \xi (x-y)} e^{-2\pi^2 \sigma^2 \xi^2} d\xi dy$$

$$\text{Now } \int e^{2\pi i \xi (x-y)} e^{-2\pi^2 \sigma^2 \xi^2} d\xi$$
$$= \mathcal{F}_\xi^{-1} [e^{-2\pi^2 \sigma^2 \xi^2}] (y-x)$$
$$= \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(y-x)^2}{2\sigma^2}} = g_\sigma(y-x)$$
$$= g_\sigma(x-y)$$

$$\text{So } (*) = \int f(y) g_\sigma(x-y) dy = f * g_\sigma(x)$$

We can show that

$$\|f * g_\sigma - f\|_1 \rightarrow 0 \text{ as } \sigma \rightarrow 0$$

$$\text{and } \lim_{\sigma \rightarrow 0} f * g_\sigma = f \text{ a.e. } \underline{\text{almost everywhere}}$$

So, if $\hat{f} \in L^1$, then things become easy!

Def. Define $\check{f}(x) := \mathcal{F}^{-1}[f(\xi)](x)$

$$= \int_{-\infty}^{\infty} f(\xi) e^{2\pi i \xi x} d\xi$$

This is the inverse Fourier transf.
of $f \in L^1$

Thm 2. If f & \hat{f} are both in L^1 ,
then $(\hat{f})^\vee = (\check{f})^\wedge = f$ a.e.

Cor. If $\hat{f} = \hat{g}$, then $f = g$ a.e.
 $\hat{f} = 0$, $f = 0$ a.e.

• Many fcn in L^1 has their F.T. in L^1 ,
but not all of the L^1 fcn have $\hat{f} \in L^1$.

For $f \in L^1$ to have $\hat{f} \in L^1$, f must be
a little smooth.

$\Rightarrow \hat{f}(\xi)$ must decay as $|\xi| \uparrow$.

e.g., if $|\hat{f}(\xi)| \leq \frac{C}{1+|\xi|^2}$ for $\exists C > 0$.
then $\hat{f} \in L^1$ (easy to show.)

If $f \in C^2$ & $f', f'' \in L^1$, then f decays as
above.

"Smoothness of a fcn
 \Leftrightarrow Decay of the Fourier
transf. at high freq."

In particular, if \hat{f} has a compact support, then it's great!

Such f is called a **band-limited** fcn.

2. The Fourier Transform on L^2

So far, we have dealt with FT on L^1 .

Simply assuming $f \in L^2(\mathbb{R})$,

$\int f(x) e^{-2\pi i \xi x} dx$ may not converge.

e.g., $f(x) = \text{sinc}(x) \in L^2$

but it is **not** in L^1 .

• We will overcome this problem as follows.

Define a subspace $\mathcal{X} \subset L^1$ s.t.

$$\mathcal{X} := \{f \in L^1 \mid \hat{f} \in L^1\}$$

For any $f \in \mathcal{X}$, both f & \hat{f} are in $BC(\mathbb{R})$

so, $\mathcal{X} \subset L^2$ (because $f \in L^1 \cap BC$

$\Rightarrow f \in L^2$, which comes from the general

Thm: $L^p \cap L^r \subset L^q$ for

set $p=1, q=2, r=\infty$. $\left(0 < p < q < r \leq \infty. \right)$
(can be proved via Hölder's ineq.)

This general thm also implies

$$\|f\|_q \leq \|f\|_p \vee \|f\|_r \\ = \max(\|f\|_p, \|f\|_r)$$

Also, you can show that \mathcal{X} is **dense** in L^2 .

- Hence, for any $f \in L^2$, we can find a sequence $\{f_n\} \subset \mathcal{X}$ s.t. $\|f_n - f\|_2 \xrightarrow{n \rightarrow \infty} 0$
 $\{f_n\} \subset \mathcal{X}$ implies $\{\hat{f}_n\} \subset \mathcal{X}$ the Plancherel eq.

Now, we can show $\|\hat{f}_n - \hat{f}_m\|_2 \stackrel{\downarrow}{=} \|f_n - f_m\|_2 \rightarrow 0$ as $n, m \rightarrow \infty$

That is, $\{f_n\}$ is a **Cauchy sequence** in L^2 .

Since L^2 is **complete**, \hat{f}_n has a limit \hat{f} in L^2 , which we define this limit as \hat{f} , the Fourier transf. of f .

Parseval is for Fourier series.

* The Plancherel Thm (\mathcal{F} is an isometry in L^2)

For any $f, g \in L^2$,

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle, \text{ hence } \|f\|_2 = \|\hat{f}\|_2$$

• Finally, we can do the following:

Suppose $\phi(x) \in L^2$. Then $\exists \hat{\phi} \in L^2$ s.t. $\phi(x) = \hat{\phi}(x)$.

Then, $\phi(\xi) = f(-\xi)$. Ex: $\phi(x) = \text{sinc}(x) \Rightarrow \hat{\phi}(\xi) = \chi_{(-\frac{1}{2}, \frac{1}{2})}(\xi)$.