Lecture 3: Uncertainty Principles

Note Title

☆ Heisenberg's Inequality (L^2)

Impossible for a signal to be both time (or space) limited and band limited.

Band-Limited:

Let $f \in L^2$, and $\hat{f}(\xi) = 0$ for $|\xi| > \frac{\Omega}{2}$

$$\Rightarrow \quad f(x) = \int_{-\Omega/2}^{\Omega/2} \hat{f}(\xi) e^{2\pi i \xi x} d\xi$$

Now consider the following fcn of $z \in \mathbb{C}$:

$$F(z) = \int_{-\Omega/2}^{\Omega/2} e^{2\pi i \xi z} \hat{f}(\xi)\, d\xi$$

This makes sense for any $z \in \mathbb{C}$, and becomes an analytic fcn in z (e.g., satisfies the Cauchy-Riemann eqn.); in fact, it is an entire fcn in $z \in \mathbb{C}$.

↳ analytic on the whole $\mathbb{C}$.

Therefore, f being a restriction of F on the real axis, f cannot vanish except at some isolated points unless $f \equiv 0$.

space-lim.: By the same token, if $f(x) = 0$ for $|x| > \frac{A}{2}$, then $\hat{f}(\xi)$ cannot vanish unless $\hat{f} \equiv 0$.

$\Rightarrow$ f cannot be localized in both domains!

Heisenberg's inequality (a.k.a. uncertainty principle) gives us more precise statement.

Def. Dispersion (or spread) of f about $x=a$

$$\Delta_a f := \int (x-a)^2 |f(x)|^2 dx \Big/ \int |f(x)|^2 dx.$$

If we define $p(x) := |f(x)|^2 / \int |f(x)|^2 dx$, then $\Delta_a f$ is the 2nd moment of p around $x=a$.
$p(x)$ can be viewed as a pdf $\leftarrow \int p(x)\, dx = 1$.

Thm (Heisenberg's Inequality)

$\forall f \in L^2$, $\quad (\Delta_{x_0} f)(\Delta_{\xi_0} \hat{f}) \geq \dfrac{1}{16\pi^2}$

for all $x_0, \xi_0 \in \mathbb{R}$.

(Pf) For technical convenience, in addition to $f \in L^2$, assume : $\begin{cases} f \in C(\mathbb{R}) \cap PS(\mathbb{R}) \\ xf \in L^2, \ f' \in L^2 \end{cases}$ piecewise smooth fcns

The reasoning behind these assumptions:

$$xf \notin L^2 \iff \int x^2 f^2(x)\, dx = \infty \iff \Delta_{x_0} f = \infty$$

$$f' \notin L^2 \iff \int |i 2\pi \xi \hat{f}(\xi)|^2 dx = \infty \iff \Delta_{\xi_0} \hat{f} = \infty$$

So, we are removing these cases from our consideration.

Let's consider first $x_0 = \xi_0 = 0$ case and consider an integral:

$$\int_a^b x \overline{f(x)} f'(x)\, dx = x |f(x)|^2 \Big|_a^b - \int_a^b \left(|f(x)|^2 + x \overline{f'(x)} f(x) \right) dx$$

Since $xf \in L^2$, $\hookrightarrow$ this goes to 0 as $\begin{cases} b \to +\infty \\ a \to -\infty \end{cases}$.

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = -2 \operatorname{Re} \int_{-\infty}^{\infty} x \overline{f(x)} f'(x)\, dx.$$

$= \|f\|_2^2$

By the Cauchy-Schwarz ineq., we have

$$\|f\|_2^4 = 4\left(\mathrm{Re}\int_{-\infty}^{\infty} \overline{x f(x)} f'(x)\,dx\right)^2$$

$$\leq 4\int_{-\infty}^{\infty} x^2|f(x)|^2 dx \cdot \int_{-\infty}^{\infty} |f'(x)|^2 dx$$

Now the Plancherel Thm tells us $\|f\|_2^2 = \|\hat{f}\|_2^2$

also, $\int |f'(x)|^2 dx = \int |\widehat{f'}(\xi)|^2 d\xi$

$$= 4\pi^2 \int \xi^2 |\hat{f}(\xi)|^2 dx$$

Hence,

$$\|f\|_2^4 = \|f\|_2^2 \cdot \|\hat{f}\|_2^2 \leq 16\pi^2 \underbrace{\int x^2|f(x)|^2 dx}_{\Delta_0 f \cdot \|f\|_2^2} \cdot \underbrace{\int \xi^2 |\hat{f}(\xi)|^2 d\xi}_{\Delta_0 \hat{f} \cdot \|\hat{f}\|_2^2}$$

$$\Longleftrightarrow \quad \Delta_0 f \cdot \Delta_0 \hat{f} \geq \frac{1}{16\pi^2}$$

General case is the same, i.e., apply the above process to $F(x) = e^{-2\pi i \xi_0 x} f(x+x_0)$.

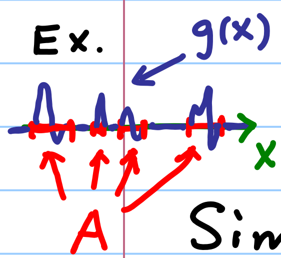
Then $\Delta_0 F = \Delta_{x_0} f$, $\Delta_0 \hat{F} = \Delta_{\xi_0} \hat{f}$

$$\Rightarrow \quad \Delta_{x_0} f \cdot \Delta_{\xi_0} \hat{f} \geq \frac{1}{16\pi^2} \qquad ///$$

Exercise : Show that the equality in Heisenberg's inequality holds iff f is a Gaussian fcn.

☆ More General Uncertainty Principles of Donoho-Stark (1989)

Rather than dealing with dispersions of f & $\hat{f}$, let's consider practical concentration of f & $\hat{f}$ on measurable sets (much more general than intervals).



Def. $f \in L^2$ is said to be ε-concentrated on a measurable set A if $\exists\, g \in L^2$ s.t. $\operatorname{supp} g = A$ and $\|f - g\|_2 \le \varepsilon$.

$\operatorname{supp} g = \overline{\{x \in \mathbb{R} \mid g(x) \neq 0\}}$

Similarly we can define the ε-concentration of $\hat{f}$.

Thm (Donoho-Stark, 1989)

Let A & Ω be measurable sets and suppose $\exists$ the Fourier transform pair $(f, \hat{f})$ with $\|f\|_2 = \|\hat{f}\|_2 = 1$ s.t. f is ε_A-concentrated on A and $\hat{f}$ is ε_Ω-concentrated on Ω.

Then,

$$|A| \cdot |\Omega| \ge \left(1 - (\varepsilon_A + \varepsilon_\Omega)\right)^2$$

In short, f & $\hat{f}$ cannot both be highly concentrated no matter what sets of concentration A & Ω we choose.

(Pf) See Donoho-Stark (1989)

This thm has a deep & surprising counterpart in the discrete setting.

Let $f = (f_0, \cdots, f_{N-1})^T$ be a vector in $\mathbb{C}^N$ and $\hat{f} = (\hat{f}_0, \cdots, \hat{f}_{N-1})^T \in \mathbb{C}^N$ be its discrete Fourier Transf. (DFT):

$$\hat{f}_k := \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} f_\ell \, e^{-2\pi i k \ell / N}, \quad k = 0, 1, \cdots, N-1.$$

(We'll cover DFT much more in our later lectures! Here, we just want to show an interesting consequence of the discrete version of the uncertainty principle.)

Let $\| f \|_0 := \# \{ \ell \in [0, N-1] \mid f_\ell \neq 0 \}$.
This is the so-called ℓ^0 quasi-norm.
A typical measure of sparsity of f.
(more precisely, $\| f \|_0$ small $\Longleftrightarrow$ f : sparse)

It's not a norm since $\| \cdot \|_0$ does not satisfy the homogeneity : $\| a f \| = |a| \cdot \| f \|, \ \forall a \in \mathbb{C}$.

Thm (Donoho–Stark, 1989)

$$\| f \|_0 \cdot \| \hat{f} \|_0 \geq N$$

→ Pf : See their article.

Cor (Donoho–Stark, 1989)

$$\| f \|_0 + \| \hat{f} \|_0 \geq 2 \sqrt{N}$$

(Pf) Easy application of AM ≥ GM. ///

Here is a more general discrete version corresponding to the continuous version.

Thm (Donoho-Stark, 1989)

Let $(\mathbf{f}, \hat{\mathbf{f}})$ be a DFT pair of unit norm with

$\mathbf{f}$: ε_A-concentrated on the index set A and
$\hat{\mathbf{f}}$: ε_Ω-concentrated on the index set Ω.
Then, $\|\mathbf{f}\|_0 \cdot \|\hat{\mathbf{f}}\|_0 \geq N(1-(\varepsilon_A+\varepsilon_\Omega))^2$.

☆ **Recovery of a sparse wide-band signal from narrow-band measurements**

As an application of the above generalized uncertainty principles, consider the following signal processing problem, often appears in practice (e.g., astronomical imaging, spectroscopy, geophysics, ...).

Suppose the discrete measurement $\mathbf{r}$ is a noisy, band-limited version of the ideal signal $\mathbf{s}$, i.e.,

$$(*) \qquad \mathbf{r} = \underbrace{P_\Omega}_{\text{BL-operator}} \mathbf{s} + \underbrace{\mathbf{n}}_{\text{noise}}$$

$$(P_\Omega \mathbf{s})_\ell := \frac{1}{\sqrt{N}} \sum_{k \in \Omega} \underbrace{\hat{s}_k}_{\text{True DFT coeff's of } \mathbf{s}} e^{2\pi i k \ell / N}$$

By taking the DFT on both sides of (*)

$$\Rightarrow \quad \hat{r}_k = \begin{cases} \hat{s}_k + \hat{n}_k & \text{if } k \in \Omega \\ 0 & \text{otherwise.} \end{cases}$$

where we also assumed $P_\Omega \mathbf{n} = \mathbf{n}$ (i.e., $\mathbf{n}$ is also band-limited).

Can we recover $\mathbf{s}$ from $\mathbf{r}$?
Enter the uncertainty principle!

Thm (Donoho-Stark, 1989)

Suppose $\mathbf{n} \equiv 0$ (no noise), and we know $\|\mathbf{s}\|_0 \leq N_0 < N$.

If $\quad N_0 \cdot (N - |\Omega|) < N/2 \qquad (\star)$

(# of missing freq. components)

then $\mathbf{s}$ can be uniquely reconstructed from $\mathbf{r}$.

$\Rightarrow$ If $\mathbf{s}$ is sparse, $\exists$ a chance of recovery!

(Pf) Uniqueness: Suppose $\mathbf{s}'$ also generates $\mathbf{r}$, i.e., $P_\Omega \mathbf{s}' = \mathbf{r} = P_\Omega \mathbf{s}$.
Define $\mathbf{h} := \mathbf{s}' - \mathbf{s}$ so that $P_\Omega \mathbf{h} = 0$.

Now $\|\mathbf{s}'\|_0 \leq N_0 \Rightarrow \|\mathbf{h}\|_0 \leq 2N_0$
Because $P_\Omega \mathbf{h} = 0$, $\|\hat{\mathbf{h}}\|_0 \leq N - |\Omega|$ $\quad \Big\} (\star\star)$

$\Rightarrow \mathbf{h} \equiv 0$ otherwise $\mathbf{h}$ violates the uncertainty principle $\|\mathbf{h}\|_0 \cdot \|\hat{\mathbf{h}}\|_0 \geq N$ because $(\star)$ & $(\star\star)$ lead to $\|\mathbf{h}\|_0 \cdot \|\hat{\mathbf{h}}\|_0 < N$.

How about the reconstruction algorithm?

Let $\tilde{s} := \arg\min_{s' \in S_0} \| r - P_\Omega s' \|$

where $S_0 := \{ f \in \mathbb{C}^N \mid \|f\|_0 \le N_0 < N \}$

From the uniqueness, we know $\tilde{s} = s$.

But how can we find such $\tilde{s}$?

$\Rightarrow$ a combinatorial algorithm.

Let $\Pi :=$ the $\binom{N}{N_0}$ subsets $\{\tau\}$ of indices $\{0, 1, \cdots, N-1\}$ with $|\tau| = N_0$.

For a given $\tau \in \Pi$, let

This is a least squares problem! $\rightarrow$ $\tilde{s}_\tau := \arg\min_{s'} \{ \| r - P_\Omega s' \| \mid \operatorname{supp} s' = \tau \}$

$\Rightarrow \exists\, \tau_0 \in \Pi$ s.t. $\tilde{s}_{\tau_0} = \tilde{s}$, i.e.,

$$\tilde{s} = \arg\min_{\tilde{s}_\tau, \tau \in \Pi} \| r - P_\Omega \tilde{s}_\tau \|.$$

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$\Rightarrow$ Impractical for large N.

$\Rightarrow \exists$ a much better approach using L^1 (or ℓ^1) compressed sensing. Read Donoho-Stark as its beginning!

How about the noisy case?

<u>Thm (Donoho-Stark, 1989)</u>

Suppose $\|s\|_0 \le N_0 < N$ with $2N_0(N - |\Omega|) < N$. Assume $\|n\| \le \varepsilon$. If $\tilde{s}$ satisfies $\|\tilde{s}\|_0 \le N_0$ and $\| r - P_\Omega \tilde{s} \| \le \varepsilon$, <u>then</u>

$$\| s - \tilde{s} \| \le 2\varepsilon \Big/ \sqrt{1 - 2N_0(1 - |\Omega|/N)}$$

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