

Lecture 5: Fourier Series on Intervals

Note Title

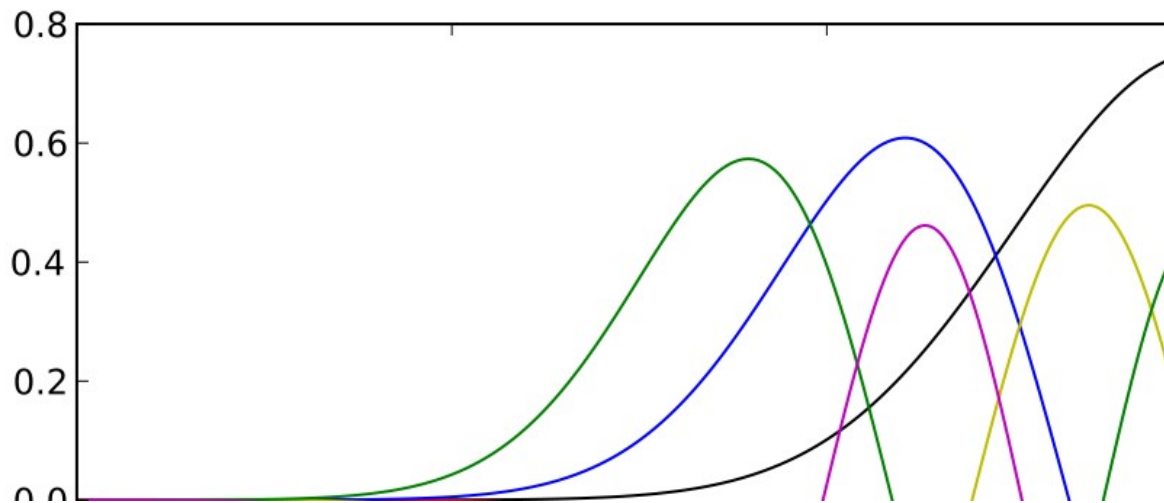
Some ways to convert an L^2 fcn into a sequence:

(1) If $\text{supp}(f)$ is compact (i.e., $\left. \begin{array}{l} \text{time} \\ \text{space} \end{array} \right\}$ -limited)
 \Rightarrow **Fourier series**

(2) If $\text{supp}(\hat{f})$ is compact (i.e., band-limited)
 \Rightarrow **Sampling**

(3) Otherwise, can use **wavelets, local cosines** etc.

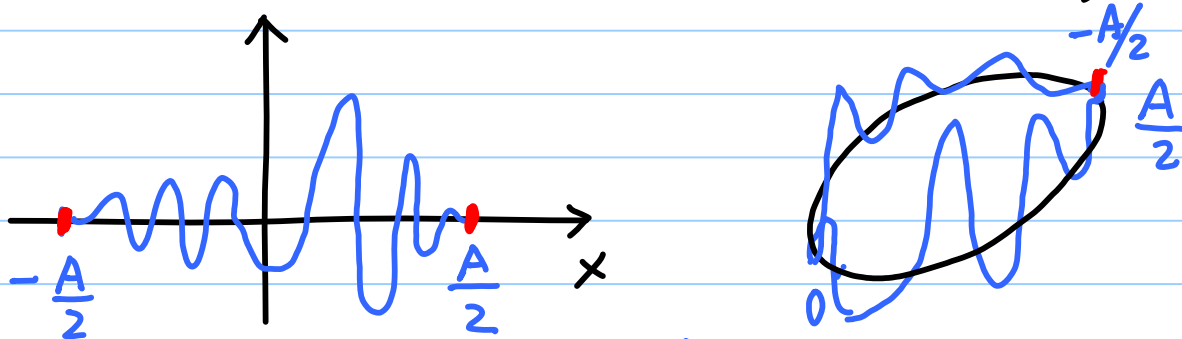
(4) Another possibility: **Hermite fcn's** (slow through)
$$\psi_n(x) = (-1)^n (2^n n! \sqrt{\pi})^{-1/2} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2}$$



(5) Other methods ...

We already discussed (2).
Now let's discuss (1): **Fourier Series!**

W.L.O.G., let's consider $f \in L^2[-\frac{A}{2}, \frac{A}{2}]$
 (since we can always shift a given fcn with a compact support to $[-\frac{A}{2}, \frac{A}{2}]$.)



Consider this as a periodic fcn with period A .
 wrap around on a circle

Thm $\left\{ \frac{1}{\sqrt{A}} e^{2\pi i k x / A} \right\}_{k \in \mathbb{Z}}$ form an ONB
 of $L^2[-\frac{A}{2}, \frac{A}{2}]$.

Note: $\frac{1}{\sqrt{A}} e^{2\pi i k x / A} = \delta_A(e^{2\pi i k x})$

So $L^2[-\frac{A}{2}, \frac{A}{2}] \cong \delta_A(L^2[-\frac{1}{2}, \frac{1}{2}])$

also note: this ONB can be written as

$$\left\{ \frac{1}{\sqrt{A}} \right\} \cup \left\{ \sqrt{\frac{2}{A}} \cos \frac{2\pi k x}{A} \right\}_{k \in \mathbb{N}} \\ \cup \left\{ \sqrt{\frac{2}{A}} \sin \frac{2\pi k x}{A} \right\}_{k \in \mathbb{N}}$$

(Pf) Let $\varphi_k(x) := \frac{1}{\sqrt{A}} e^{2\pi i k x / A}$, $k \in \mathbb{Z}$.

Then it's easy to show $\{\varphi_k\}_{k \in \mathbb{Z}}$ is an ON set, i.e., $\langle \varphi_k, \varphi_l \rangle = \delta_{kl}$. ✓

The main issue is to prove its **completeness**. Kronecker's delta

Below is a sketch/idea of the proof.

Define

$$C_{\#}(-A/2, A/2) := \{f \in C(-A/2, A/2) \mid f(-A/2) = f(A/2)\}$$

$$C_0(-A/2, A/2) := \{f \in C(-A/2, A/2) \mid f(x) = 0 \text{ for } x \in \exists \underbrace{N(-A/2) \cup N(A/2)}_{\text{neighborhood of } \pm A/2}\}$$

Then, clearly,

$$C_0 \subset C_{\#} \subset L^2$$

Things to show:

Step 1: C_0 is **dense** in L^2 , hence $C_{\#}$ is also **dense** in L^2 .

Step 2: Let $\mathcal{M} := \overline{\text{span}\{\varphi_k\}}$

want to show $\mathcal{M} = L^2$.

But, it suffices to show $C_{\#} \subset \mathcal{M}$ thanks to Step 1.

To do so, one needs to show:

$$\forall f \in C_{\#}, f \in \mathcal{M}.$$

See, e.g., Dym & McKean (Chap. 1) for the details. //

Now, we can safely write for $f \in L^2[-\frac{A}{2}, \frac{A}{2}]$,

$$\begin{aligned} f &= \sum_{-\infty}^{\infty} \alpha_k \varphi_k, \quad \alpha_k = \langle f, \varphi_k \rangle \\ &= \frac{1}{\sqrt{A}} \int_{-\frac{A}{2}}^{\frac{A}{2}} f(x) e^{-2\pi i k x / A} dx \\ &= \hat{f}(k) \end{aligned}$$

This notation, $\hat{f}(k)$, is justified considering the dual of the important "take-home" idea of Lecture 4:

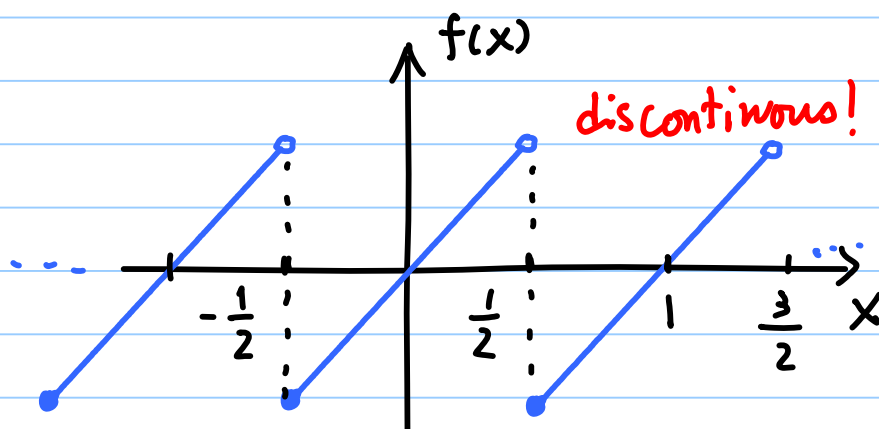
"Periodization in $\left\{ \begin{array}{l} \text{space} \\ \text{time} \end{array} \right\}$

\Leftrightarrow Discretization in frequency."

See also my Note I, page 2.

* Examples of the Fourier Series:

(1) $f(x) = x$, $-\frac{1}{2} \leq x < \frac{1}{2}$, i.e., $A=1$.
 $\varphi_k(x) = e^{2\pi i k x}$



For $k \neq 0$, we have

$$\begin{aligned} \alpha_k &= \int_{-\frac{1}{2}}^{\frac{1}{2}} x e^{-2\pi i k x} dx = \frac{x e^{-2\pi i k x}}{-2\pi i k} \Big|_{-\frac{1}{2}}^{\frac{1}{2}} + \frac{1}{2\pi i k} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i k x} dx \\ &= \frac{\cos \pi k}{-2\pi i k} + \frac{1}{(4\pi k)^2} e^{-2\pi i k x} \Big|_{-\frac{1}{2}}^{\frac{1}{2}} \\ &= \frac{(-1)^k i}{2\pi k} \end{aligned}$$

$$\alpha_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} x dx = 0.$$

Hence $f(x) = x$ is 1-periodic $\sim \sum' \frac{(-1)^k i}{2\pi k} e^{2\pi i k x}$

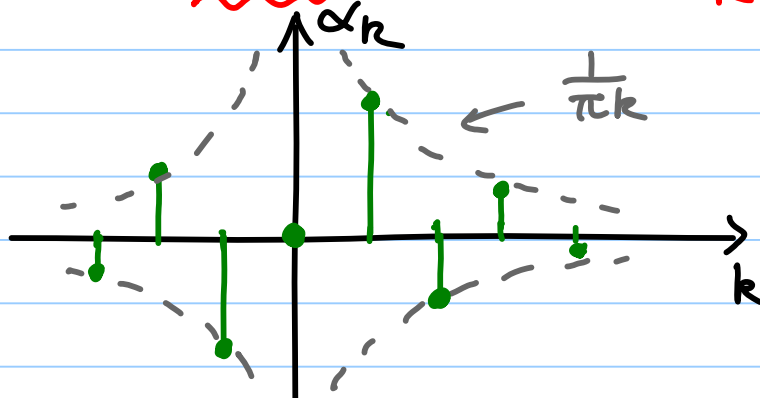
$\leftarrow k=0$ is excluded.

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\pi k} \sin(2\pi k x)$$

Remarks:

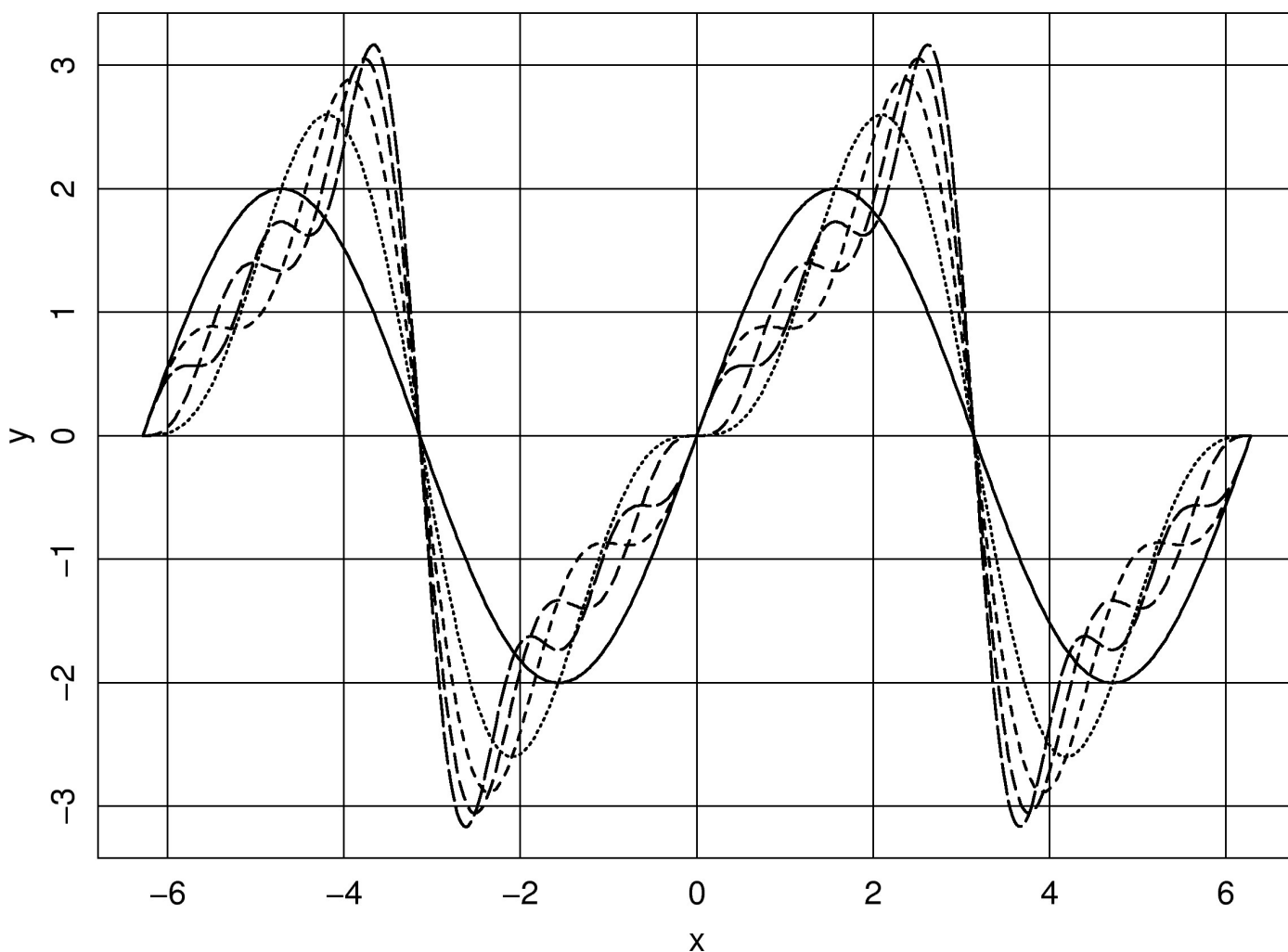
(1) $f(x) = x$ is an odd fcn.
Its Fourier series has only sine terms.
 \rightarrow makes sense.

(2) The decay of the Fourier coefficients is rather slow, i.e., $O(\frac{1}{k})$.



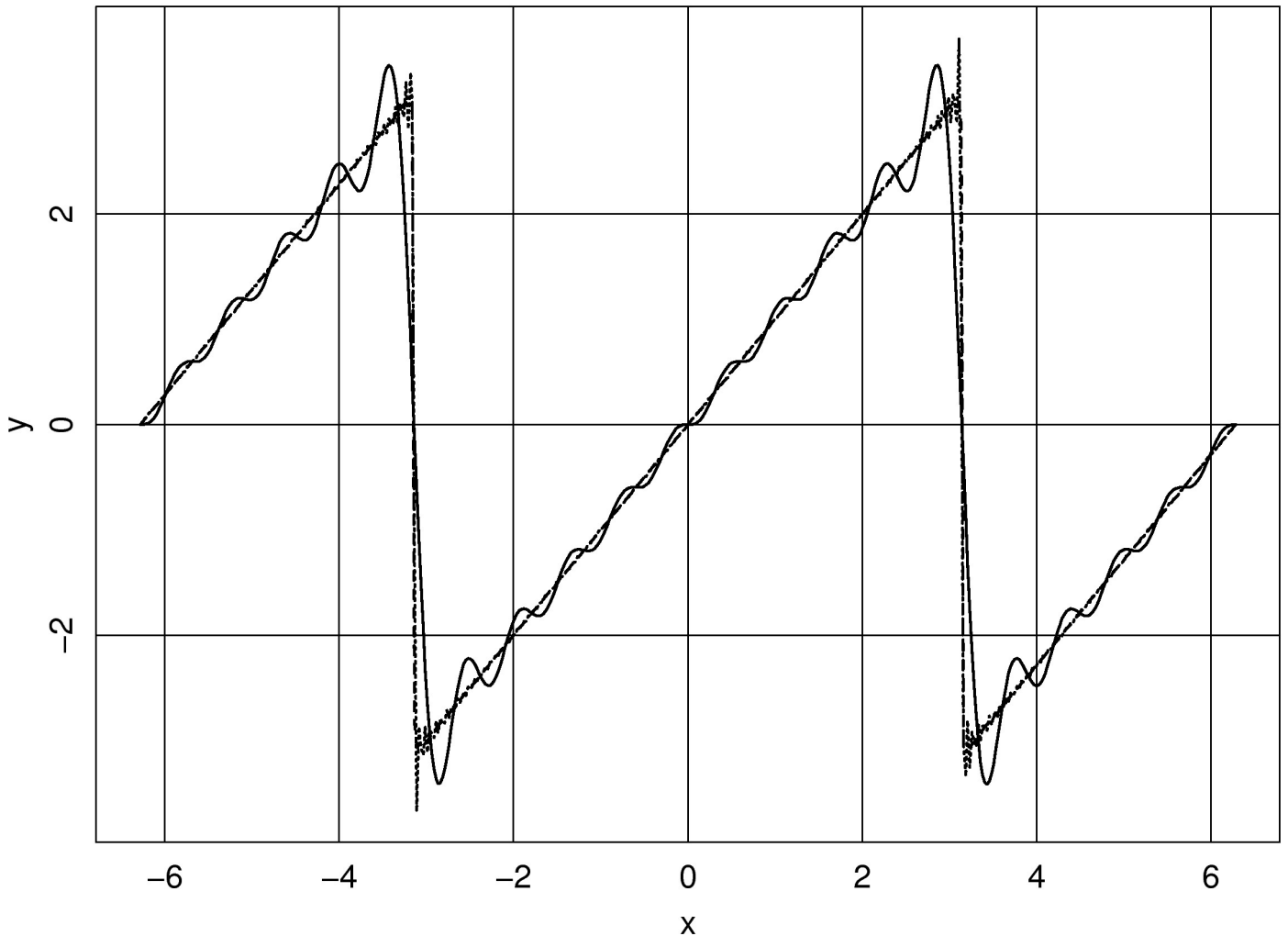
Let's look at the first few partial sums. (The interval was $[-\pi, \pi)$ instead of $[-\frac{1}{2}, \frac{1}{2})$.)

First 5 Partial Sums of the Fourier Series of $f(x)=x$ (2π period)



How about more terms?

Partial Sums: $N=10, 100, 1000$

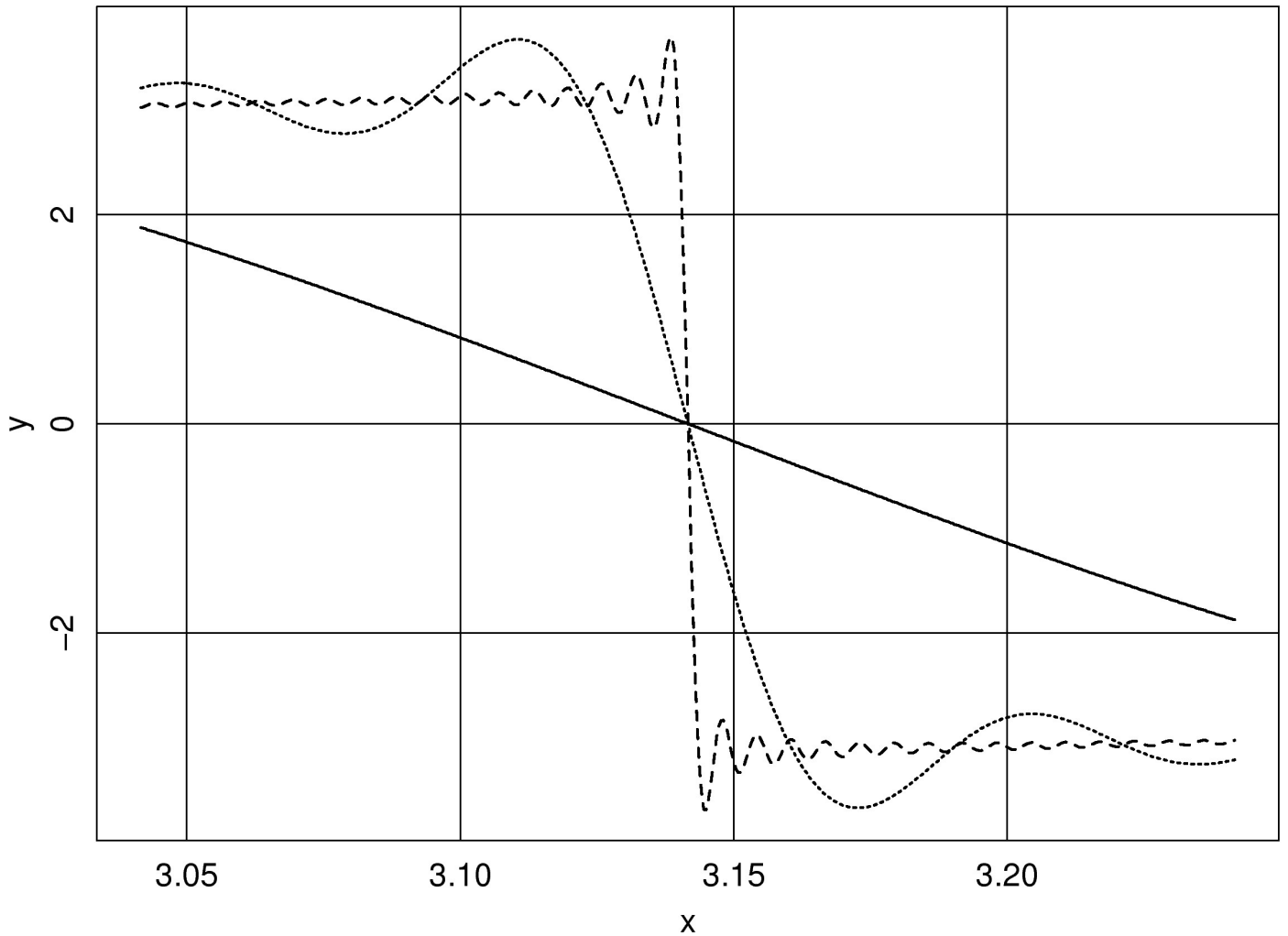


We still see the spurious oscillations around discontinuities.

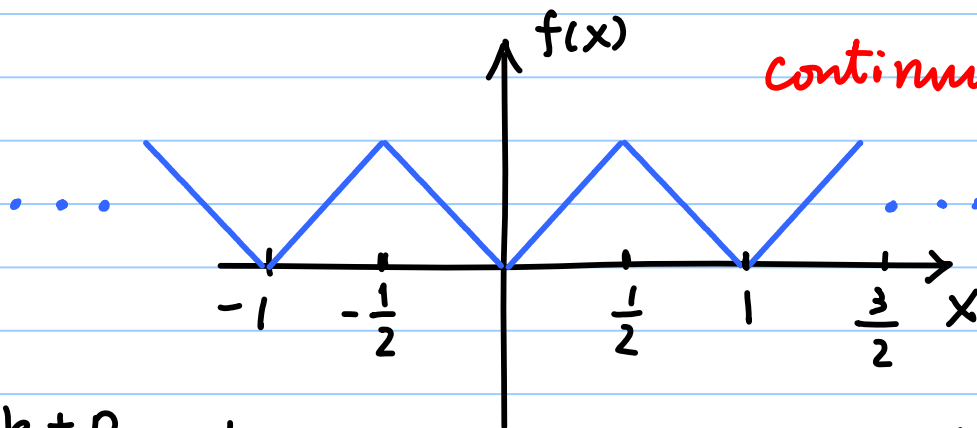
\Rightarrow This leads to the so-called Gibbs phenomenon.
(See Hewitt - Hewitt, 1979)

Zooming up around the discontinuity:

Zoom around $x=\pi$ of Partial Sums: $N=10$, 100 , 1000



$$(2) f(x) = |x|, \quad -\frac{1}{2} \leq x < \frac{1}{2}.$$



continuous!

in C
but not
in C' .

$$\text{For } k \neq 0, \quad \alpha_k = \int_{-\frac{1}{2}}^{\frac{1}{2}} |x| e^{-2\pi i k x} dx = 2 \int_0^{\frac{1}{2}} x \cos 2\pi k x dx$$

$$= 2 \left\{ x \cdot \frac{\sin 2\pi k x}{2\pi k} \Big|_0^{\frac{1}{2}} - \frac{1}{2\pi k} \int_0^{\frac{1}{2}} \sin 2\pi k x dx \right\}$$

$$= -\frac{1}{\pi k} \cdot \frac{-\cos 2\pi k x}{2\pi k} \Big|_0^{\frac{1}{2}} = \frac{\cos(\pi k) - 1}{2(\pi k)^2}$$

$$= \frac{(-1)^k - 1}{2\pi^2 k^2}$$

$$\alpha_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |x| dx = 2 \int_0^{\frac{1}{2}} x dx = 2 \cdot \frac{x^2}{2} \Big|_0^{\frac{1}{2}} = \frac{1}{4}$$

$$\text{So, } f(x) = |x| \sim \frac{1}{4} - \frac{1}{2} \sum'_{k \in \mathbb{Z}} \frac{1 - (-1)^k}{\pi^2 k^2} e^{2\pi i k x}$$

1-periodic

$$= \frac{1}{4} - \sum_1^{\infty} \frac{1 - (-1)^k}{\pi^2 k^2} \cos(2\pi k x)$$

$$= \frac{1}{4} - 2 \sum_{k=1}^{\infty} \frac{\cos(2\pi(2k-1)x)}{\pi^2 (2k-1)^2}$$

Remarks:

(1) The decay of the Fourier coeff. is faster: $O\left(\frac{1}{k^2}\right)$

(2) Evaluating this at $x=0$ leads to

$$f(0) = 0 = \frac{1}{4} - 2 \sum_{k=1}^{\infty} \frac{1}{\pi^2 (2k-1)^2}$$

$$\Leftrightarrow \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$

So, we can also prove the celebrated

Basel problem: $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$

$$(Pf) \quad \sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \left(\frac{1}{(2k-1)^2} + \frac{1}{(2k)^2} \right)$$

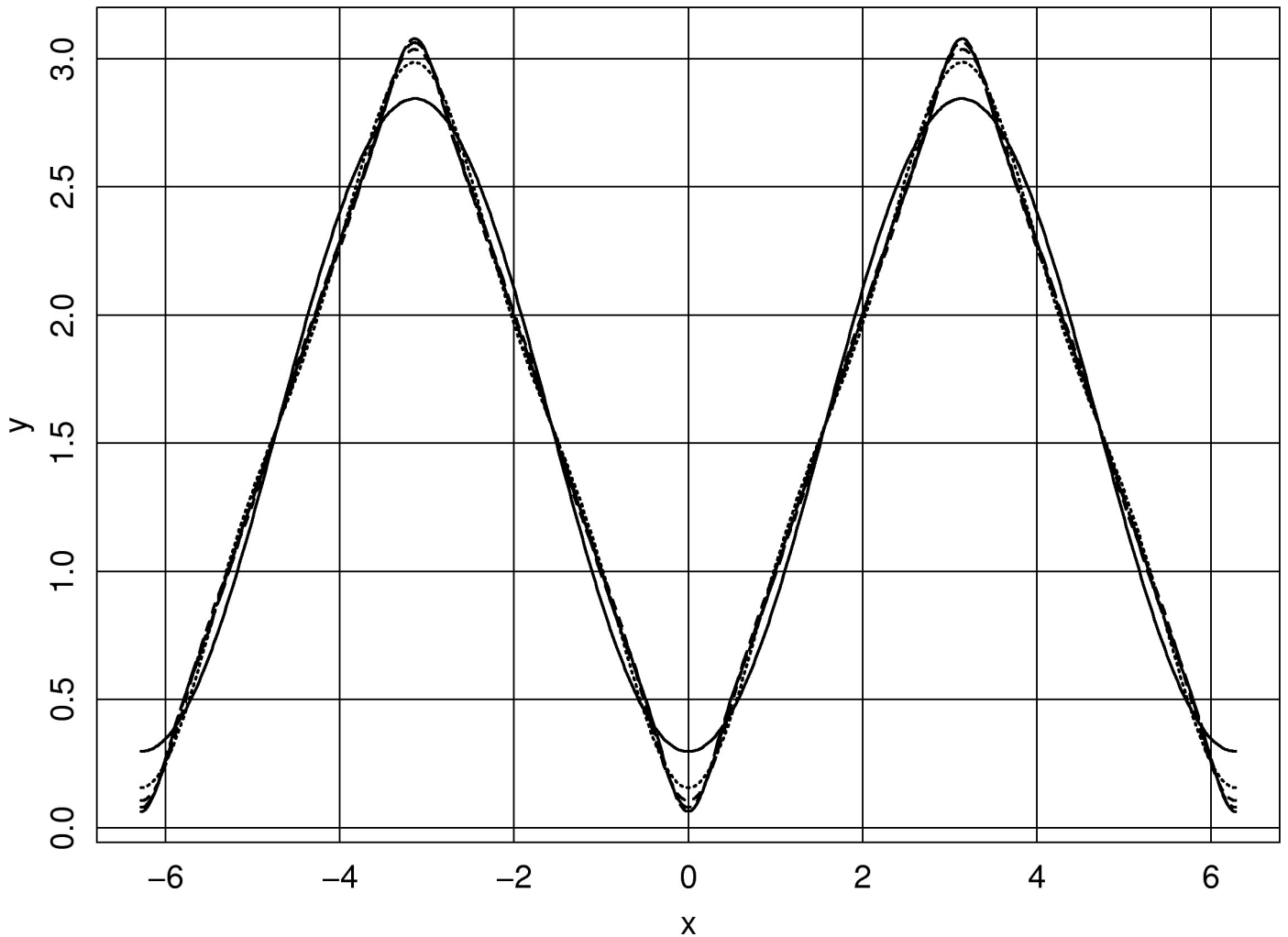
$$= \frac{\pi^2}{8} + \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$\Leftrightarrow \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \quad \equiv \equiv \equiv$$

\exists 8 or so ways to prove the Basel problem !!

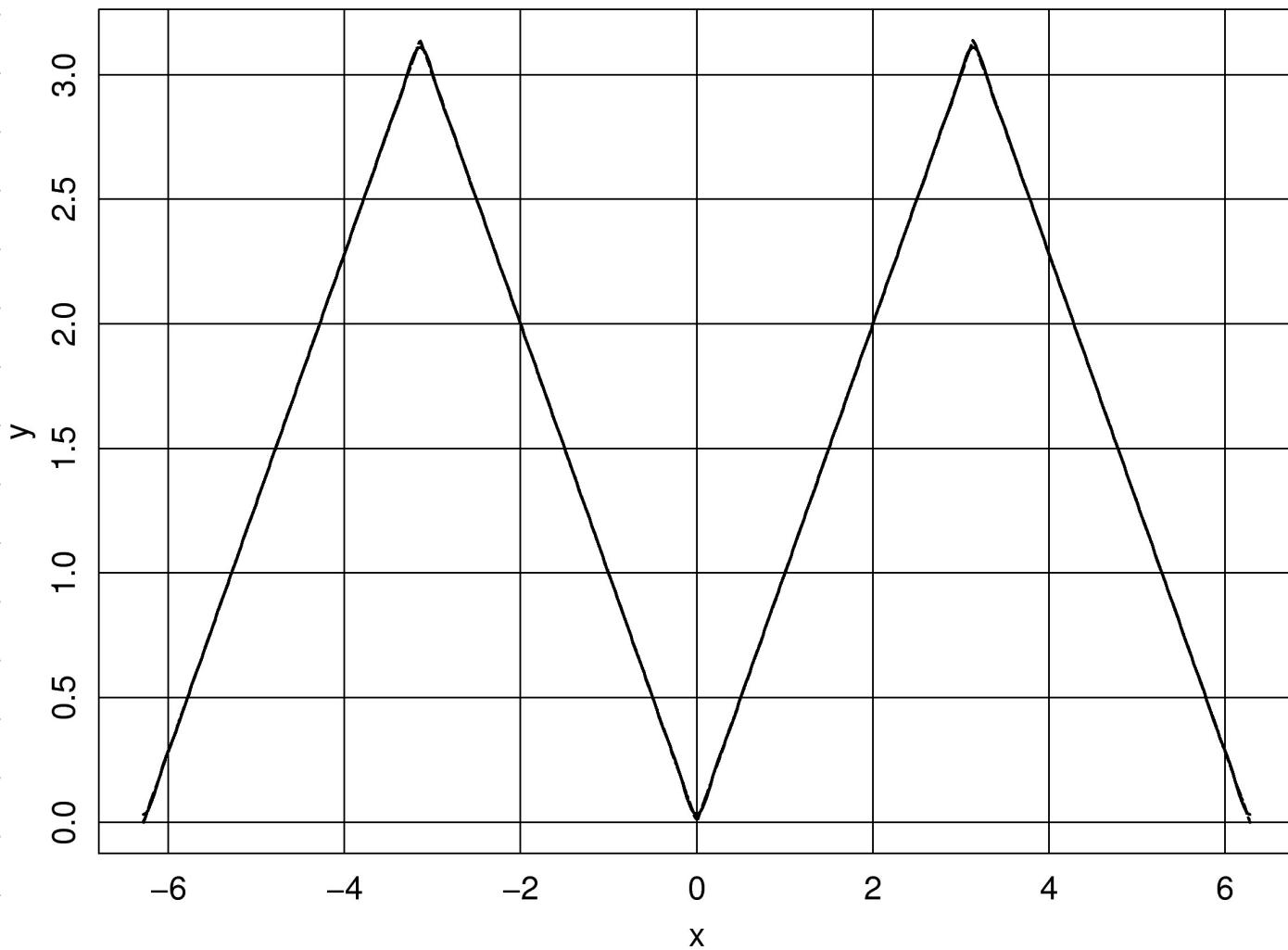
How good is it to have faster decaying Fourier coefficients?

First 5 Partial Sums of the Fourier Series of $f(x)=|x|$ (2π period)



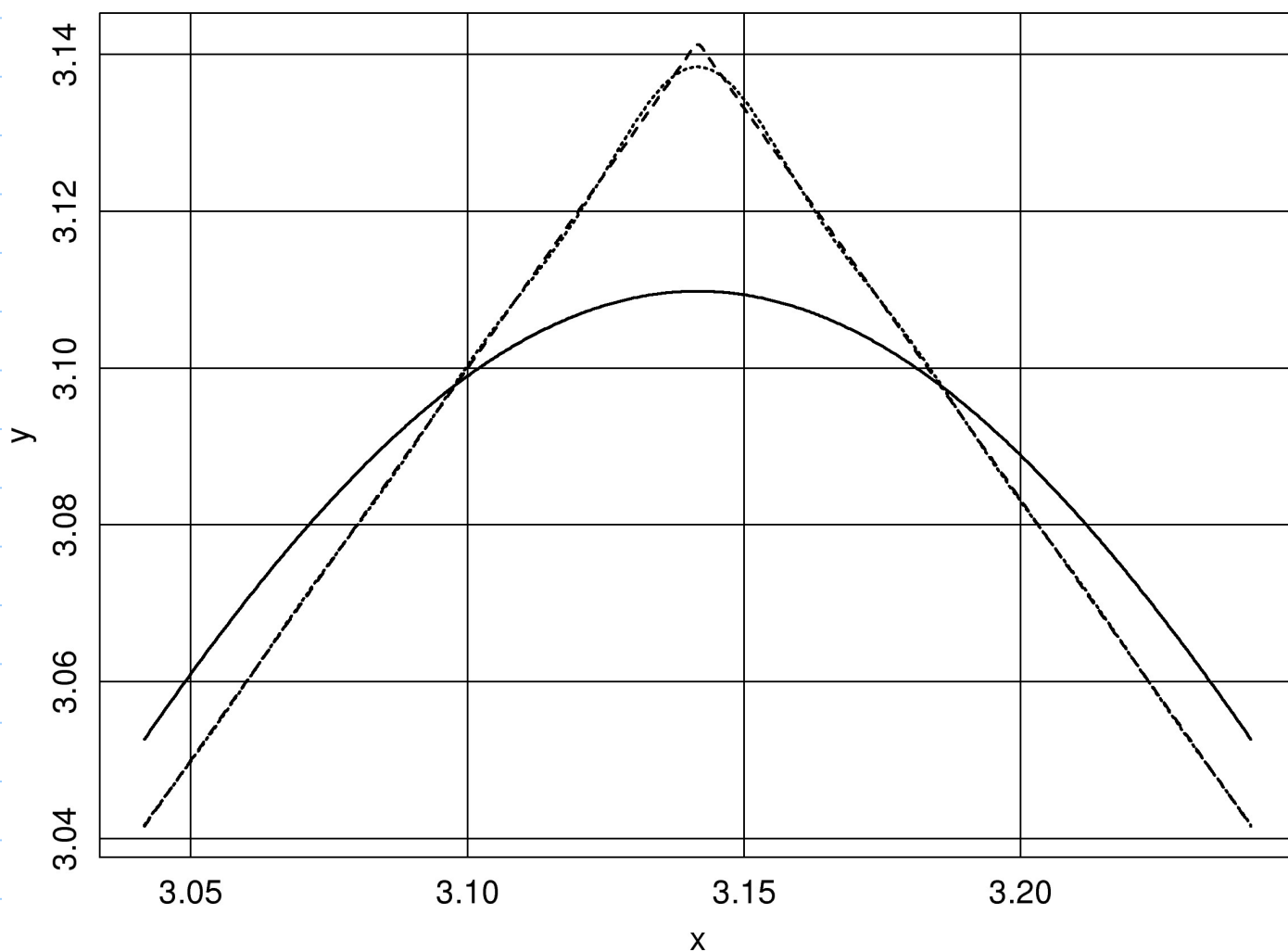
More terms?

Partial Sums: $N=10, 100, 1000$



Let's zoom up at the apex at $x = \pi$ where f' is discontinuous (but f is cont.)

Zoom around $x = \pi$ of Partial Sums: $N=10, 100, 1000$



MAT 271: Applied & Computational Harmonic Analysis: Supplementary Notes II by Naoki Saito

A Brief History of the Convergence of the Fourier Series

Theorem 1 (Dirichlet, 1829) Suppose f is 1-periodic, piecewise smooth on \mathbb{R} . Then, n th partial sum,

$$S_n[f](x) := \sum_{-n}^n c_k e^{2\pi i k x}, \text{ satisfies}$$

$$\lim_{n \rightarrow \infty} S_n[f](x) = \frac{1}{2} [f(x+) + f(x-)].$$

In particular, if x is a point of continuity, then $\lim_{n \rightarrow \infty} S_n[f](x) = f(x)$.

Theorem 2 (du Bois Reymond, 1876) There exists $f \in C(I)$ such that $\{S_n[f](0)\}$ diverges, where I is an interval of unit length.

Theorem 3 (A weak version of Fejér's Theorem) If f is 1-periodic, *continuous*, and piecewise smooth on \mathbb{R} , then the Fourier series of f converges to f *absolutely* and *uniformly*.

Definition: Suppose a series of functions $\sum_1^\infty g_n(x)$ converges to $g(x)$ on a set $x \in I$. Then, the convergence is called *absolute* if $\sum_1^\infty |g_n(x)|$ also converges for $x \in I$.

If we have $\sup_{x \in I} \left| g(x) - \sum_1^N g_n(x) \right| \rightarrow 0$ as $N \rightarrow \infty$, then we call this a *uniform* convergence.

Theorem 4 (Fejér 1904) If $f \in C(I)$, then the Cesàro means of $S_n[f]$ converge *uniformly* to f .

Definition: The m th *Cesàro mean* of partial sums is the mean of the first $m + 1$ partial sums, i.e.,

$$\sigma_m[f](x) := \frac{1}{m+1} \sum_{n=0}^m S_n[f](x).$$

Theorem 5 (Size of the Fourier coefficients and the smoothness of the functions) Suppose f is 1-periodic. If $f \in C^{k-1}(\mathbb{R})$ and $f^{(k-1)}$ is piecewise smooth (i.e., $f^{(k)}$ exists and piecewise continuous), then the Fourier coefficients of f , c_n , satisfy $\sum_n |n^k c_n|^2 < \infty$. In particular, $n^k c_n \rightarrow 0$. On the other hand, suppose $c_n, n \neq 0$, satisfy $|c_n| \leq C|n|^{-(k+\gamma)}$ for some $C > 0$ and $\gamma > 1$. Then $f \in C^k(\mathbb{R})$.

Theorem 6 (Kolmogorov, 1926) There exists $f \in L^1(I)$ such that $\{S_n[f](x)\}$ diverges for every x .

Theorem 7 (Carleson, 1966) If $f \in L^2(I)$, then $S_n[f](x)$ converges to $f(x)$ almost everywhere.

Theorem 8 (Hunt, 1967) If $f \in L^p(I), p > 1$, then $S_n[f](x)$ converges to $f(x)$ almost everywhere.

Mathematicians are still trying to simplify the proof of the Carlson-Hunt theorem as of today.

For the details of the above facts, see [1, Chap. 1,2], [2, Chap. 1], [3, Part 1], and [4, Chap. 1]. [5, Chap. 1].

References

- [1] J. M. ASH, ed., *Studies in Harmonic Analysis*, vol. 13 of MAA Studies in Mathematics, Math. Assoc. Amer., 1976.

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- [3] T. W. KÖRNER, *Fourier Analysis*, Cambridge Univ. Press, 1988.
- [4] S. G. KRANTZ, *A Panorama of Harmonic Analysis*, no. 27 in The Carus Mathematical Monographs, Math. Assoc. Amer., Washington, D.C., 1999.
- [5] M. A. PINSKY, *Introduction to Fourier Analysis and Wavelets*, Amer. Math. Soc., Providence, RI, 2002. Republished by AMS, 2009.

★ Smoothness Class Hierarchy

(from Davis & Rabinowitz:

"Methods of Numerical Integration"
Sec. 1.9, Dover 2007.)

From rough to smooth on an interval $[a, b]$
within the class of Riemann-integrable fcn's.

- $R[a, b]$: bdd. & Riemann-integrable
- $BV[a, b]$: bdd. variation
- $PC[a, b]$: Piecewise-continuous
- $C[a, b]$: continuous
- $Lip_\alpha[a, b]$: Lipschitz (or Hölder) continuous $0 < \alpha \leq 1$.
- $C^1[a, b]$: continuously differentiable
- $C^n[a, b]$: n times cont. diff.
- $A(\Omega)$, $[a, b] \subset \Omega \subset \mathbb{C}$: analytic on Ω
- $E(\mathbb{C})$: entire (i.e., analytic on \mathbb{C})

Def. Lipschitz (or Hölder) continuity.

$$Lip_\alpha[a, b] := \left\{ f \in C[a, b] \mid |f(x) - f(y)| \leq K \cdot |x - y|^\alpha, \right. \\ \left. \equiv K \geq 0, \forall x, y \in [a, b] \right\}.$$

Ex. $f(x) = \sqrt{x} \notin Lip_\alpha[0, 1]$ with $\frac{1}{2} < \alpha \leq 1$, but
 $\in Lip_\alpha[0, 1]$ with $0 < \alpha \leq \frac{1}{2}$

In the next lecture, we will discuss more
about $BV[a, b]$, fcn's of bdd. variation!