

Lecture 11: Time-Frequency Analysis/Synthesis

Note Title

- * Now we are in a good position to discuss how to beat the Heisenberg uncertainty principle.
- * Shortcomings of the Fourier basis:
 - Too **global** in space (or time)
 - Not good for **edges, singularities** in signals
 - Not good for **nonstationary** signals, i.e., signals changing their characteristics in time.

Our Roadmap

- * (General) **time-frequency atoms**
 - * **Windowed (or short-time) Fourier transf.**
 - { Boxcar window
 - { Gaussian window → **Gabor fens**
 - Local cosine transf. (LCT)
 - * **Wavelet transforms**
 - { **Continuous** wavelet transf. (**redundant**)
 - { **discrete** wavelet transf.
 - { **orthonormal**
 - { **biorthogonal**
 - redundant**
 - wavelet packets
- **Frame Theory**
-

★ Time-Frequency Atoms

$\{\phi_\gamma\}_{\gamma \in \Gamma} \subset L^2(\mathbb{R})$, Γ : some multiindex set

$\|\phi_\gamma\|_2 = 1$. e.g., $\{(m, n)\}_{(m, n) \in \mathbb{Z}^2}$
 m : time index, n : freq. index

The correlation of a given fcn $f \in L^2(\mathbb{R})$ with $\{\phi_\gamma\}_{\gamma \in \Gamma}$ can be measured by

$$Tf(\gamma) := \int_{-\infty}^{\infty} f(x) \overline{\phi_\gamma(x)} dx = \langle f, \phi_\gamma \rangle$$

By Plancherel's equality, $\langle f, \phi_\gamma \rangle = \langle \hat{f}, \hat{\phi}_\gamma \rangle$

Recall the Heisenberg uncertainty principle

$$\Delta_{x_0}^2 f \Delta_{\xi_0}^2 \hat{f} \geq \frac{1}{16\pi^2} \quad \forall x_0, \xi_0 \in \mathbb{R}.$$

where $\Delta_{x_0}^2 f := \int (x-x_0)^2 |f(x)|^2 dx / \|f\|_2^2$

Suppose $m_x(\phi_\gamma) := \int x |\phi_\gamma(x)|^2 dx$.

the **center of gravity** of $|\phi_\gamma|^2$

Similarly $m_\xi(\hat{\phi}_\gamma) := \int \xi |\hat{\phi}_\gamma(\xi)|^2 d\xi$

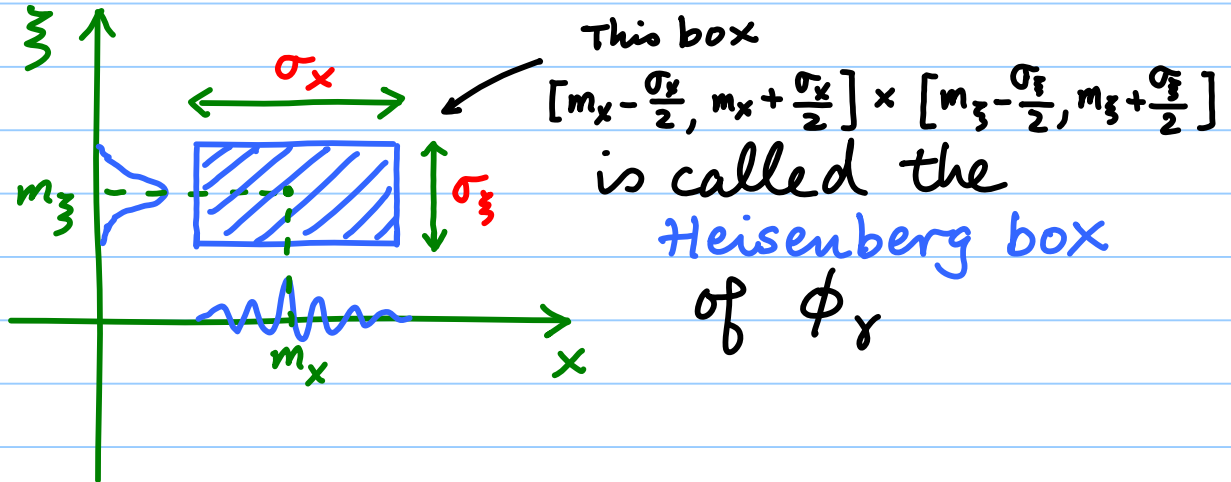
Since $\|\phi_\gamma\|_2 = \|\hat{\phi}_\gamma\|_2 = 1$, we have

$$\Delta_{m_x}^2 \phi_\gamma \cdot \Delta_{m_\xi}^2 \hat{\phi}_\gamma \geq \frac{1}{16\pi^2}$$

By defining $\begin{cases} \sigma_x := \sqrt{\Delta_{m_x}^2 \phi_\gamma} \\ \sigma_\xi := \sqrt{\Delta_{m_\xi}^2 \hat{\phi}_\gamma} \end{cases}$

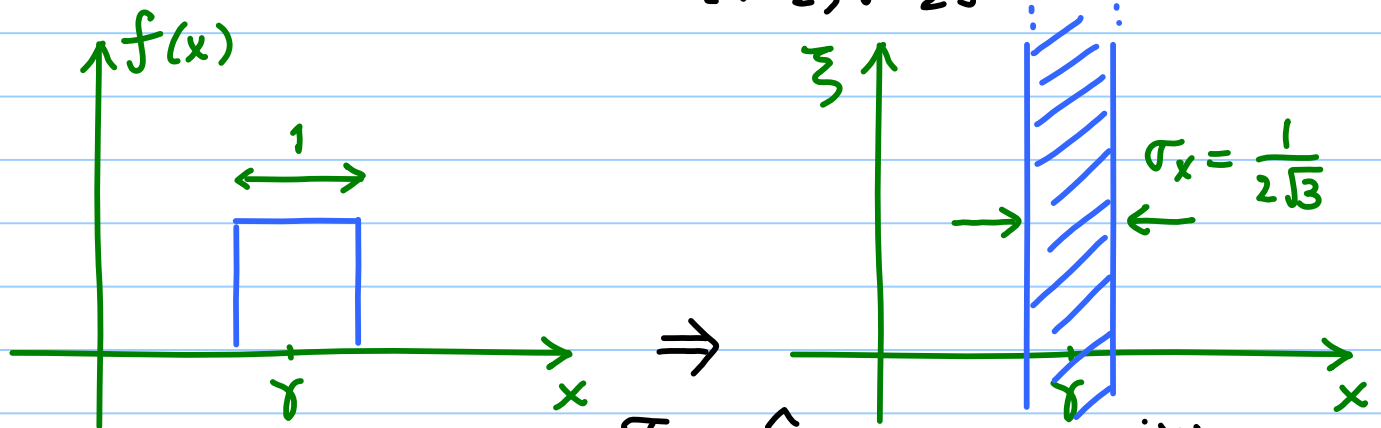
the Heisenberg inequality can be written as

$$\sigma_x \sigma_\xi \geq \frac{1}{4\pi}$$



Note that the main energy of ϕ_γ is in this box but not all the energy in it.

Example $\phi_\gamma(x) = \chi_{[\gamma-\frac{1}{2}, \gamma+\frac{1}{2}]}(x)$



$$\phi_\gamma(x) = \tau_\gamma \chi_{[-\frac{1}{2}, \frac{1}{2}]}(x) \xrightarrow{\mathcal{F}} \hat{\phi}_\gamma(\xi) = e^{-2\pi i \gamma \xi} \text{sinc}(\xi)$$

$$m_x = \int_{\gamma-\frac{1}{2}}^{\gamma+\frac{1}{2}} x \cdot 1 \, dx = \gamma$$

$$m_\xi = \int_{-\infty}^{\infty} \xi \cdot \text{sinc}^2(\xi) \, d\xi = 0$$

odd (above the integral) and *even* (below the integral)

$$\Delta_{m_x}^2 \phi_\gamma = \int_{\gamma-\frac{1}{2}}^{\gamma+\frac{1}{2}} (x-\gamma)^2 \, dx = \frac{1}{12}$$

$$\Delta_{m_\xi}^2 \hat{\phi}_\gamma = \int_{-\infty}^{\infty} \xi^2 \cdot \frac{\text{sin}^2 \pi \xi}{\pi^2 \xi^2} \, d\xi = +\infty$$

$\sigma_x = 1/2\sqrt{3}$ but $\sigma_\xi = \infty \Rightarrow$ **No frequency resolution!**

Use different families of atoms

(1) Windowed Fourier Atoms \rightarrow Windowed (short-time) Fourier transf.
 (aka. Time-Frequency Atoms)

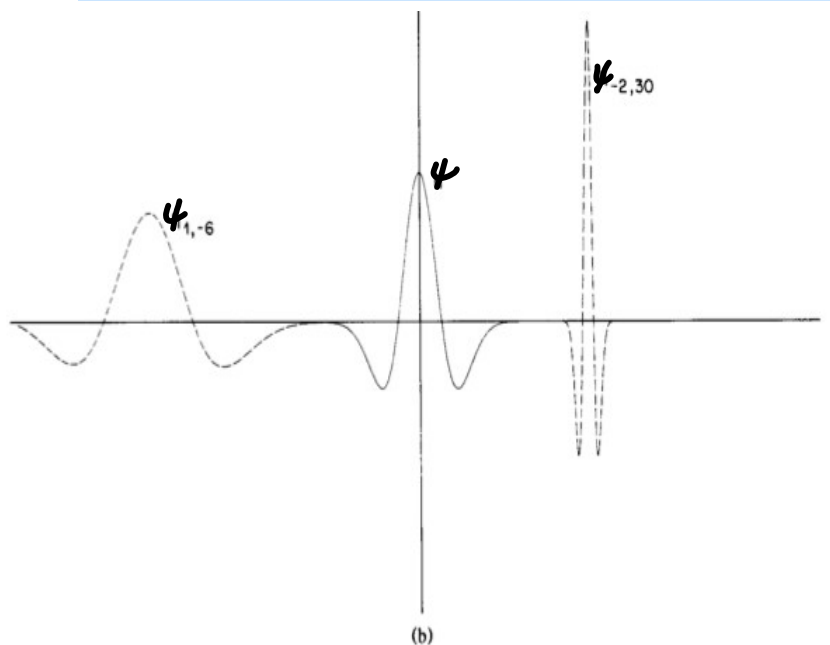
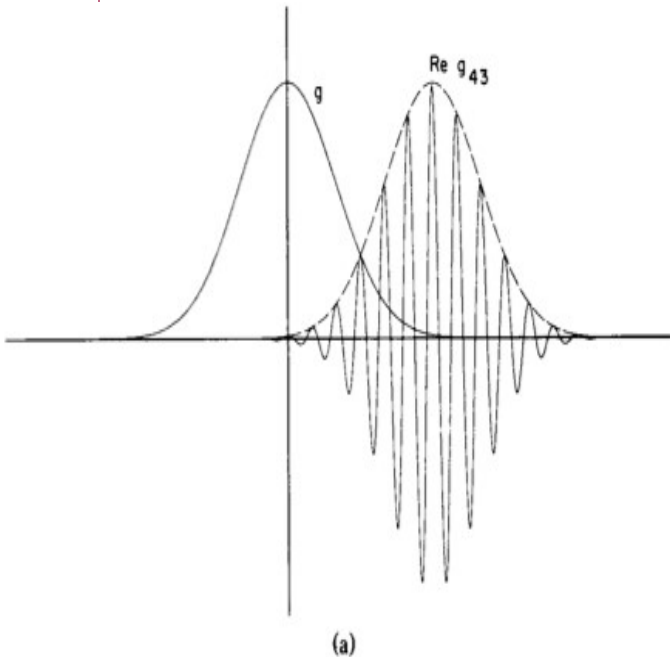
$$\phi_{\gamma}(x) = g_{x_0, \xi_0}(x) := e^{2\pi i \xi_0 x} g(x - x_0)$$

g : some window fcn. modulation translation
 e.g., Gaussian. translation in the freq. dom.

(2) Wavelet Atoms \rightarrow Wavelet transf.
 (aka. Time-Scale Atoms)

$$\phi_{\gamma}(x) = \psi_{a,b}(x) := \tau_b \delta_a \psi(x) = \frac{1}{\sqrt{a}} \psi\left(\frac{x-b}{a}\right)$$

ψ : a "mother" wavelet \Rightarrow must satisfy some conditions.



Windowed Fourier Atoms

Wavelet Atoms

(1) Windowed Fourier Transforms

$$g_{x_0, \xi_0}(x) = e^{2\pi i \xi_0 x} g(x - x_0)$$

assume $\begin{cases} \|g\|_2 = 1 \text{ so that } \|g_{x_0, \xi_0}\|_2 = 1, \\ g: \text{real-valued \& even} \end{cases}$

For $f \in L^2(\mathbb{R})$, define

$$\begin{aligned} Sf(x_0, \xi_0) &:= \langle f, g_{x_0, \xi_0} \rangle = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi_0 x} \overline{g(x - x_0)} dx \\ &= \int_{-\infty}^{\infty} f(x) g(x - x_0) e^{-2\pi i \xi_0 x} dx \\ &= \mathcal{F}[f \cdot \tau_{x_0} g](\xi_0) \end{aligned}$$

checking f around (x_0, ξ_0) on the t - f plane.

How good g_{x_0, ξ_0} is in terms of the Heisenberg box?

$$\begin{aligned} m_x(g_{x_0, \xi_0}) &= \int_{-\infty}^{\infty} x |g_{x_0, \xi_0}(x)|^2 dx = \int_{-\infty}^{\infty} x g^2(x - x_0) dx \\ &= \int_{-\infty}^{\infty} (y + x_0) g^2(y) dy \quad \leftarrow y = x - x_0 \\ &= \int_{-\infty}^{\infty} y g^2(y) dy + x_0 \int_{-\infty}^{\infty} g^2(y) dy = x_0 \end{aligned}$$

odd $\int_{-\infty}^{\infty} y g^2(y) dy = 0$
 $\int_{-\infty}^{\infty} g^2(y) dy = \|g\|_2^2 = 1$

$$\hat{g}_{x_0, \xi_0}(\xi) = \mathcal{F}[e^{2\pi i \xi_0 x} \cdot \tau_{x_0} g](\xi)$$

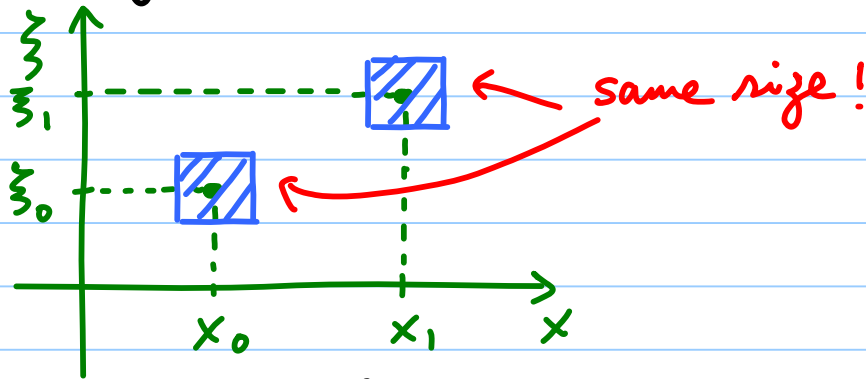
$$= \tau_{\xi_0}(e^{-2\pi i \xi x_0} \hat{g}(\xi))$$

$$= e^{-2\pi i (\xi - \xi_0) x_0} \hat{g}(\xi - \xi_0)$$

$$\begin{aligned} \Rightarrow m_{\xi_0}(\hat{g}_{x_0, \xi_0}) &= \int_{-\infty}^{\infty} \xi |\hat{g}_{x_0, \xi_0}(\xi)|^2 d\xi \\ &= \int_{-\infty}^{\infty} \xi |\hat{g}(\xi - \xi_0)|^2 d\xi = \xi_0 \end{aligned}$$

$$\begin{cases} \Delta_{m_x}^2 g_{x_0, \xi_0} = \int_{-\infty}^{\infty} (x-x_0)^2 |g(x-x_0)|^2 dx = \int_{-\infty}^{\infty} x^2 g^2(x) dx \\ \Delta_{m_\xi}^2 \hat{g}_{x_0, \xi_0} = \int_{-\infty}^{\infty} (\xi-\xi_0)^2 |\hat{g}(\xi-\xi_0)|^2 d\xi = \int_{-\infty}^{\infty} \xi^2 |\hat{g}(\xi)|^2 d\xi \end{cases}$$

These quantities are completely specified by g only, independent from (x_0, ξ_0)



If $g(x) = \frac{1}{\sqrt{4\pi}} e^{-x^2/2}$, then $\|g\|_2 = 1$ and $\hat{g}(\xi) = \sqrt{2} \sqrt{\pi} e^{-2\pi^2 \xi^2}$. Hence, we have

$$\Delta_{m_x}^2 g_{x_0, \xi_0} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{1}{2}$$

$$\begin{aligned} \Delta_{m_\xi}^2 \hat{g}_{x_0, \xi_0} &= 2\sqrt{\pi} \int_{-\infty}^{\infty} \xi^2 e^{-4\pi^2 \xi^2} d\xi = 2\sqrt{\pi} \int_{-\infty}^{\infty} \left(\frac{y}{2\pi}\right)^2 e^{-y^2} \frac{dy}{2\pi} \\ &= 2\sqrt{\pi} \frac{1}{8\pi^3} \cdot \frac{\sqrt{\pi}}{2} = \frac{1}{8\pi^2} \end{aligned}$$

$$\Rightarrow \Delta_{m_x}^2 g_{x_0, \xi_0} \cdot \Delta_{m_\xi}^2 \hat{g}_{x_0, \xi_0} = \frac{1}{16\pi^2}$$

achieving the lower bound in the Heisenberg inequality!

How about using $g_\sigma(x)$ instead of $g(x)$?

Note that our previous definition of g_σ was

$$g_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}, \quad \|g_\sigma\|_1 = 1$$

But we want here $\|g_\sigma\|_2 = 1$.

So, we redefine it by $g_\sigma(x) := \frac{1}{4\sqrt{\pi}\sqrt{\sigma}} e^{-x^2/2\sigma^2}$

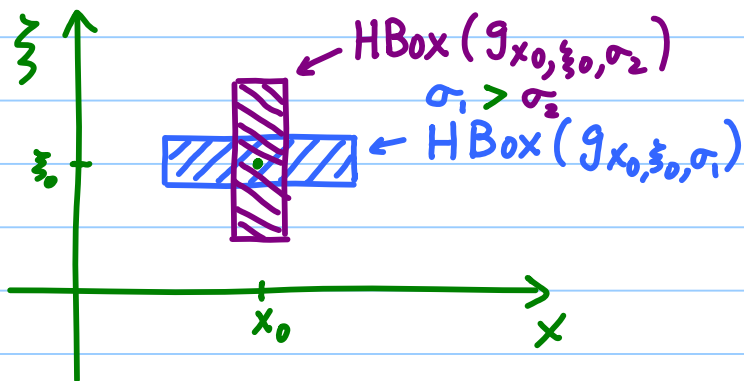
We can show:

$$m_x(g_{x_0, \xi_0, \sigma}) = x_0, \quad m_\xi(\hat{g}_{x_0, \xi_0, \sigma}) = \xi_0$$

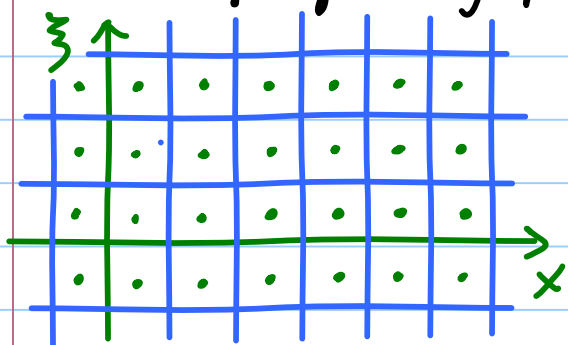
$$\Delta_{m_x}^2 g_{x_0, \xi_0, \sigma} = \frac{\sigma^2}{2}, \quad \Delta_{m_\xi}^2 \hat{g}_{x_0, \xi_0, \sigma} = \frac{1}{8\pi^2 \sigma^2}$$

$$\Rightarrow \Delta_{m_x}^2 g_{x_0, \xi_0, \sigma} \cdot \Delta_{m_\xi}^2 \hat{g}_{x_0, \xi_0, \sigma} = \frac{1}{16\pi^2}$$

Still achieves the lower b.d.

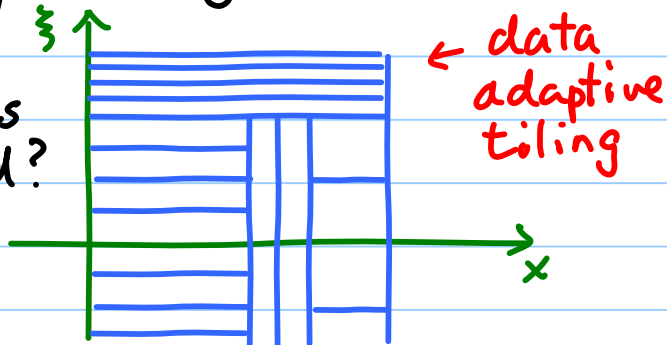


With a constant σ , we can "tile" the time-frequency plane by rectangular boxes.



can we do this instead?

⇒



• Show the **Wavelab** demo here!

So far, we have talked only "analysis".
How about "synthesis" or "representation"?

Thm If $f \in L^2(\mathbb{R})$, then

$$f(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Sf(y, \eta) g_{y, \eta}(x) dy d\eta,$$

$$\text{and } \|f\|_2^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Sf(x, \xi)|^2 dx d\xi$$

$$\text{i.e., } Sf \in L^2(\mathbb{R}^2).$$

Note that WFT is very **redundant**!
 $S: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}^2)$

(Pf) First of all, let's compute

$$\mathcal{F}_x [Sf(x, \eta)] = \widehat{Sf}(\xi, \eta). \text{ To do so,}$$

$$\begin{aligned} Sf(x, \eta) &= \int_{-\infty}^{\infty} f(y) g(y-x) e^{-2\pi i \eta y} dy \\ &= e^{-2\pi i \eta x} \int_{-\infty}^{\infty} f(y) g(y-x) e^{-2\pi i \eta (y-x)} dy \\ &\stackrel{\substack{g(y-x) = g(x-y) \\ \text{since } g: \text{even}}}{=} e^{-2\pi i \eta x} \int_{-\infty}^{\infty} f(y) g(x-y) e^{2\pi i \eta (x-y)} dy \\ &= e^{-2\pi i \eta x} \cdot f * g_{0, \eta}(x) \\ &\quad \downarrow \mathcal{F}_x \end{aligned}$$

$$\begin{aligned} (*) \widehat{Sf}(\xi, \eta) &= \widehat{f}(\xi + \eta) \cdot \widehat{g}_{0, \eta}(\xi + \eta) = \widehat{f}(\xi + \eta) \widehat{g}(\xi) \\ \text{because } g_{0, \eta}(x) &= g(x) e^{2\pi i \eta x} \rightarrow \widehat{g}_{0, \eta}(\xi) = \widehat{g}(\xi - \eta) \end{aligned}$$

Now,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S f(y, \eta) g_{y, \eta}(x) dy d\eta$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} S f(y, \eta) \underbrace{g(x-y)}_{= g(y-x) = \tau_x g(y)} dy \right\} e^{2\pi i \eta x} d\eta$$

$$= \langle S f(\cdot, \eta), \tau_x g(\cdot) \rangle$$

Plancherel \rightarrow

$$= \langle \widehat{S f}(\cdot, \eta), e^{-2\pi i x \cdot} \widehat{g}(\cdot) \rangle$$

via (*) \rightarrow

$$= \int_{-\infty}^{\infty} \widehat{f}(\xi + \eta) \widehat{g}(\xi) e^{2\pi i \xi x} \overline{\widehat{g}(\xi)} d\xi$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widehat{f}(\xi + \eta) |\widehat{g}(\xi)|^2 e^{2\pi i (\xi + \eta) x} d\xi d\eta$$

Fubini \rightarrow

$$= \int_{-\infty}^{\infty} |\widehat{g}(\xi)|^2 \left\{ \int_{-\infty}^{\infty} \widehat{f}(\xi + \eta) e^{2\pi i (\xi + \eta) x} d\eta \right\} d\xi$$

= $f(x)$ inverse FT!

$$= f(x) \int_{-\infty}^{\infty} |\widehat{g}(\xi)|^2 d\xi = f(x) \|\widehat{g}\|_2^2 = f(x) \cdot$$

= $\|g\|_2^2 = 1$

Because the WFT is redundant, it is **not** true that any $\Phi \in L^2(\mathbb{R}^2)$ is a WFT of some $f \in L^2(\mathbb{R})$, i.e., not "onto".

Prop. Let $\Phi \in L^2(\mathbb{R}^2)$. Then,

$$\exists f \in L^2(\mathbb{R}) \text{ s.t. } \Phi(x, \xi) = S f(x, \xi)$$

$$\Leftrightarrow \underbrace{\Phi(x_0, \xi_0)} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underbrace{\Phi(x, \xi)} K(x_0, x, \xi_0, \xi) dx d\xi$$

where $\underbrace{K(x_0, x, \xi_0, \xi)} := \langle g_{x, \xi}, g_{x_0, \xi_0} \rangle$

(Pf) Exercise! a **reproducing** kernel