Lecture 13: Wavelet Transforms

Note Title

* **Problems of STFT & Gabor systems**
  - If a window is too large (wide), then cannot localize around sharp transitions in an input signal.
  - If a window is too small (narrow), then cannot detect low freq. oscillations.
  - The Balian-Low Thm: $\nexists$ "nice" Gabor ONBs

* **Key idea of wavelets:**
  Use translations and dilations of a single fcn to analyze a given signal at different resolutions.

Def. A wavelet is a fcn $\psi \in L^2(\mathbb{R})$ s.t.

often called a "mother" wavelet

- $\int_{-\infty}^{\infty} \psi(x)\, dx = 0$;
- Normalized to have $\|\psi\| = 1$; and
- Centered around $x = 0$.

Let's generate a family of TF atoms:

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\, \psi\left(\frac{x-b}{a}\right)$$

$$= \tau_b\, \delta_a\, \psi \qquad a > 0,\ b \in \mathbb{R}$$

Note $\|\psi_{a,b}\|_2 = 1$.

The wavelet transform of $f \in L^2(\mathbb{R})$ is defined as

$$Wf(a,b) = W_\psi f(a,b) := \langle f, \psi_{a,b} \rangle$$

$$= \int_{-\infty}^{\infty} f(x) \frac{1}{\sqrt{a}} \overline{\psi\left(\frac{x-b}{a}\right)}\, dx$$

often called the "continuous" wavelet transf. (CWT)

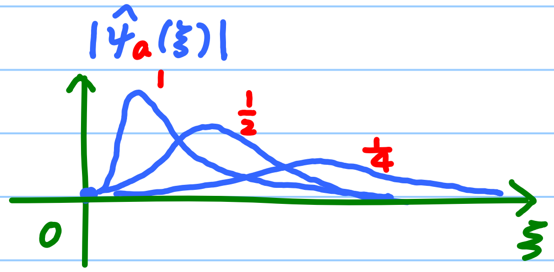
Can be viewed as a linear filtering:

$$\int_{-\infty}^{\infty} f(x) \frac{1}{\sqrt{a}} \overline{\psi\left(\frac{x-b}{a}\right)}\, dx = f \underset{\text{conv.}}{*} \tilde{\psi}_a(b)$$

$$\tilde{\psi}_a(x) := \frac{1}{\sqrt{a}} \psi\left(\frac{-x}{a}\right) \overset{\mathcal{F}}{\longmapsto} \hat{\tilde{\psi}}_a(\xi) = \sqrt{a}\, \overline{\hat{\psi}(a\xi)}$$

$$= \delta_{1/a} \overline{\hat{\psi}(\xi)}$$

mirror of $\psi_a(x)$



Types of wavelets :

- Real wavelets $\rightarrow$ Good for edges
- Analytic (or complex) wavelets $\rightarrow$ can detect phases

For the time being, let's focus on real wavelets.

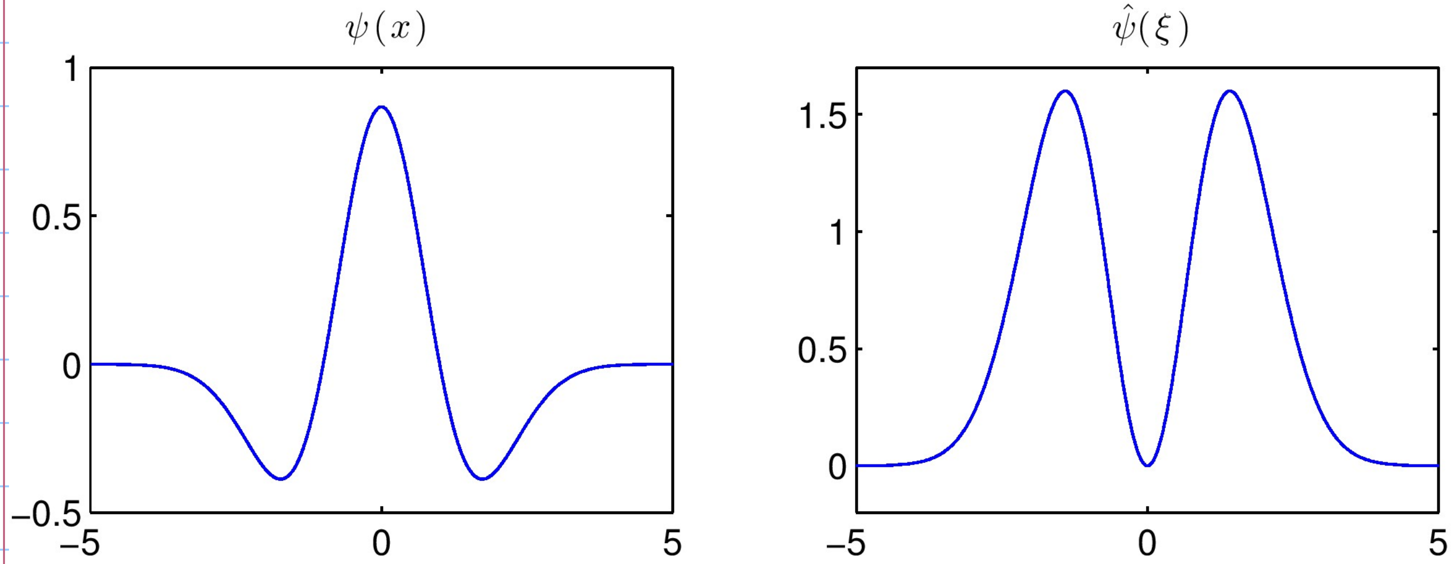
Example : Mexican hat fcn or a.k.a. Laplacian of Gaussian (LoG)

$$\begin{cases} \psi(x) = \dfrac{2}{\pi^{\frac{1}{4}}\sqrt{3\sigma}} \left(1 - \dfrac{x^2}{\sigma^2}\right) e^{-x^2/2\sigma^2} \\ \hat{\psi}(\xi) = 8\sqrt{\dfrac{2}{3}}\, \pi^{\frac{9}{4}} \sigma^{\frac{5}{2}} \xi^2 e^{-2\pi^2\sigma^2\xi^2} \end{cases}$$

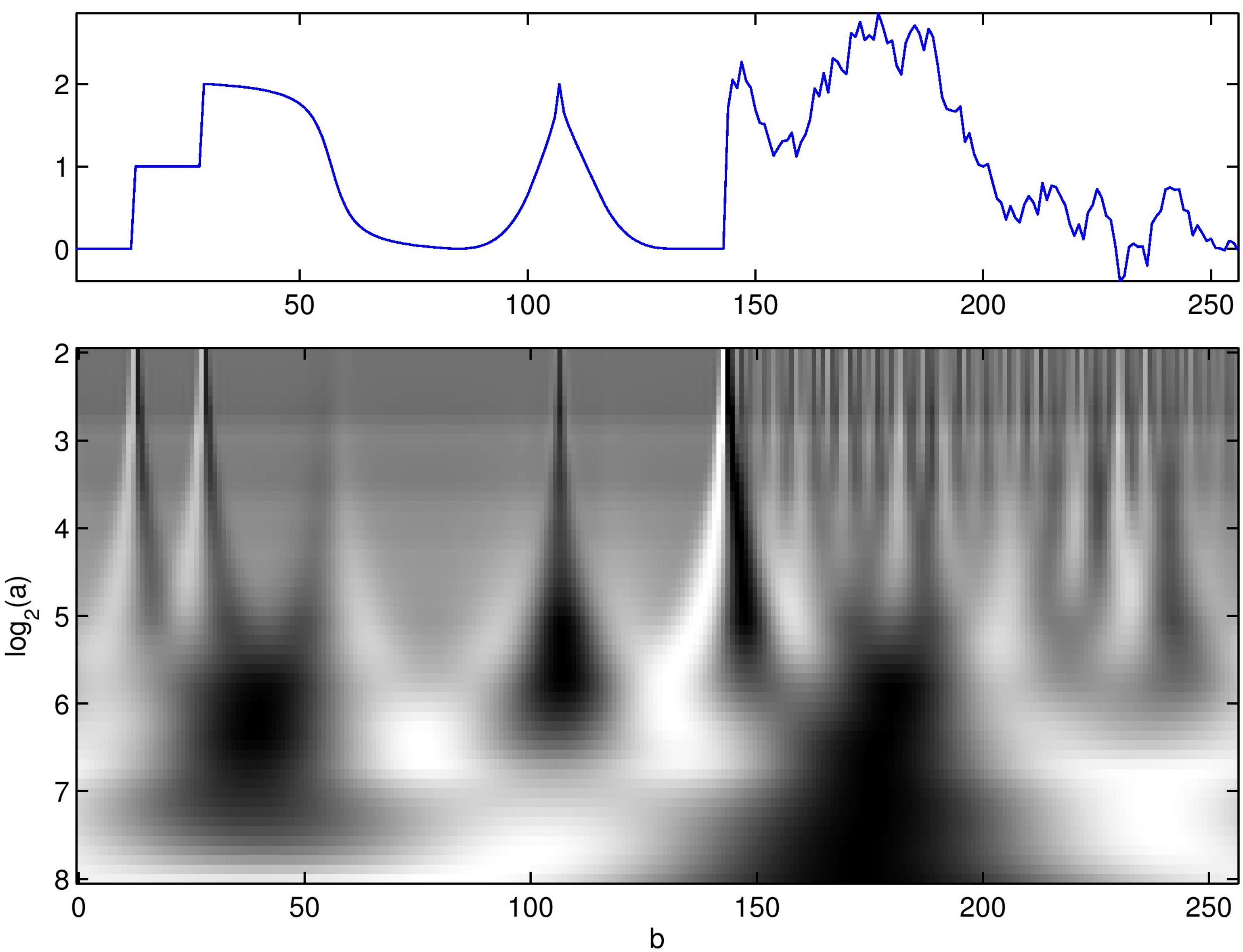
$\hat{\psi}(0) = 0, \quad \hat{\psi}(\xi) \sim \xi^2$ around $\xi = 0$

$\rightarrow$ approx. to $\frac{d^2}{dx^2}$

often called a pseudo differential op.



$\psi(x)$
$\hat{\psi}(\xi)$



log2(a)
b

Inverse Wavelet Transform

1964 1984

Thm (Calderón - Grossmann - Morlet)

Let $\psi \in L^2(\mathbb{R})$, $\psi \in \mathbb{R}$ s.t.

$$C_\psi := \int_0^\infty \frac{|\hat{\psi}(\xi)|^2}{\xi} \, d\xi < +\infty$$

Then any $f \in L^2(\mathbb{R})$ satisfies

$$(*) \quad f(x) = \frac{1}{C_\psi} \int_0^\infty \int_{-\infty}^{+\infty} Wf(a,b)\, \psi_{a,b}(x) \, db \, \frac{da}{a^2}$$

and

$$\|f\|_2^2 = \frac{1}{C_\psi} \int_0^\infty \int_{-\infty}^{\infty} |Wf(a,b)|^2 \, db \, \frac{da}{a^2}.$$

(Pf) $\quad Wf(a,b) = f * \tilde{\psi}_a(b)$

$$\text{RHS of } (*) = \frac{1}{C_\psi} \int_0^\infty \left(Wf(a,\cdot) * \psi_{a,\cdot} \right)(x) \, \frac{da}{a^2}$$

$$= \frac{1}{C_\psi} \int_0^\infty \left(f * \tilde{\psi}_a * \psi_a \right)(x) \, \frac{da}{a^2}$$

$$\Big\downarrow \mathcal{F}$$

$$= \frac{1}{C_\psi} \int_0^\infty \hat{f}(\xi) \sqrt{a}\, \overline{\hat{\psi}(a\xi)} \sqrt{a}\, \hat{\psi}(a\xi) \, \frac{da}{a^2}$$

$$= \frac{\hat{f}(\xi)}{C_\psi} \int_0^\infty \frac{|\hat{\psi}(a\xi)|^2}{a} \, da$$

$$a\xi = \eta$$

$$= \frac{\hat{f}(\xi)}{C_\psi} \underbrace{\int_0^\infty \frac{|\hat{\psi}(\eta)|^2}{\eta} \, d\eta}_{= C_\psi} = \hat{f}(\xi) \quad ///$$

$C_\psi < +\infty$ is called the admissibility condition,

$(*)$ is called Calderón's reproducing formula.

$$f(x) = \frac{1}{C_\psi} \int_0^\infty f * \tilde{\psi}_a * \psi_a(x) \, \frac{da}{a^2}$$

↳ also called the resolution of identity.

To guarantee $C_\psi < \infty$, we need

$\hat{\psi}(0) = 0 \iff \int_{-\infty}^{\infty} \psi(x)\, dx = 0$

So, ψ must be oscillatory with $\pm$ values

Also need decay on ψ

e.g., $\int_{-\infty}^{\infty} (1+|x|)\, \psi(x)\, dx < \infty$.

☆ Reproducing Kernel

CWT = a redundant representation

(**) $$Wf(a,b) = \int_{-\infty}^{\infty} \underbrace{\left(\frac{1}{C_\psi} \int_0^{\infty} \int_{-\infty}^{\infty} Wf(a',b')\, \psi_{a',b'}(x)\, db' \frac{da'}{a'^2} \right)}_{= f(x)} \overline{\psi_{a,b}(x)}\, dx$$

$$= \frac{1}{C_\psi} \int_0^{\infty} \int_{-\infty}^{\infty} K(a,a',b,b')\, Wf(a',b')\, db' \frac{da'}{a'^2}$$

where $K(a,a',b,b') := \langle \psi_{a,b}, \psi_{a',b'} \rangle$

↪ measuring the correlation between $\psi_{a,b}$ & $\psi_{a',b'}$

If $K(a,a',b,b') = \delta(a-a')\,\delta(b-b')$

then no redundancy!

Prop. a function $\Phi(a,b) \in L^2(\mathbb{R}_+ \times \mathbb{R})$ is a wavelet transform of some $f \in L^2(\mathbb{R})$

$\iff$ $\Phi(a,b)$ satisfies (**).

☆ Scaling Function ("Father" Wavelet)

Reconstruction formula requires all values of scale $0 < a < +\infty$

If we only know $Wf(a,b)$ for $a < a_0$, then we need complementary info. for $a > a_0$ provided by the scaling function (father wavelet) $\phi(x)$ s.t.

$$|\hat{\phi}(\xi)|^2 := \int_1^\infty |\hat{\psi}(a\xi)|^2 \frac{da}{a}$$

$$= \int_\xi^\infty \frac{|\hat{\psi}(\eta)|^2}{\eta} d\eta$$

The phase of ϕ can be arbitrary chosen.

- $\lim_{\xi \to 0} |\hat{\phi}(\xi)|^2 = C_\psi$
- $\|\phi\|_2 = 1 \quad \leftarrow$ Exercise, use the def.

So, the low freq. approx. of f at scale a can be written as

$$Lf(a,x) := \langle f, \overbrace{\delta_a \phi}^{\equiv \phi_a} \rangle = f * \tilde{\phi}_a(x)$$

$$\Rightarrow \quad f(x) = \frac{1}{C_\psi} \int_0^{a_0} (Wf(a,\cdot) * \psi_a)(x) \frac{da}{a^2}$$

$$\underbrace{+ \frac{1}{C_\psi a_0} (Lf(a_0,\cdot) * \phi_{a_0})(x)}_{\text{Complementary info}}$$

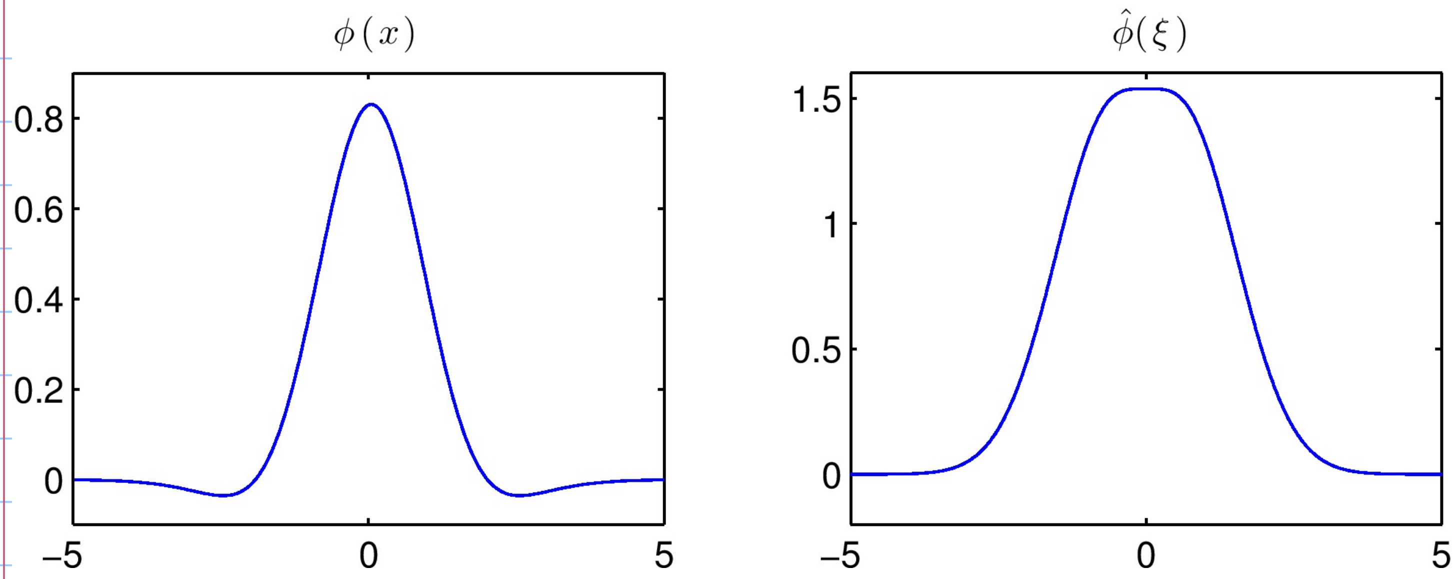
Ex. $$\psi(x) = \frac{2}{\pi^{\frac{1}{4}}\sqrt{3\sigma}}\left(1 - \frac{x^2}{\sigma^2}\right) e^{-x^2/2\sigma^2}$$

$$\hat{\psi}(\xi) = 8\sqrt{\frac{2}{3}}\,\pi^{\frac{9}{4}}\sigma^{\frac{5}{2}}\,\xi^2 e^{-2\pi^2\sigma^2\xi^2}$$

$$\Rightarrow \quad |\hat{\phi}(\xi)|^2 = \frac{4\sigma}{3\sqrt{\pi}}\left(1 + 4\pi^2\sigma^2\xi^2\right) e^{-4\pi^2\sigma^2\xi^2}$$

$$\Rightarrow \quad \hat{\phi}(\xi) = 2\sqrt{\frac{\sigma}{3\sqrt{\pi}}}\sqrt{1+4\pi^2\sigma^2\xi^2}\; e^{-2\pi^2\sigma^2\xi^2}$$

↳ choose a simple phase factor. ///



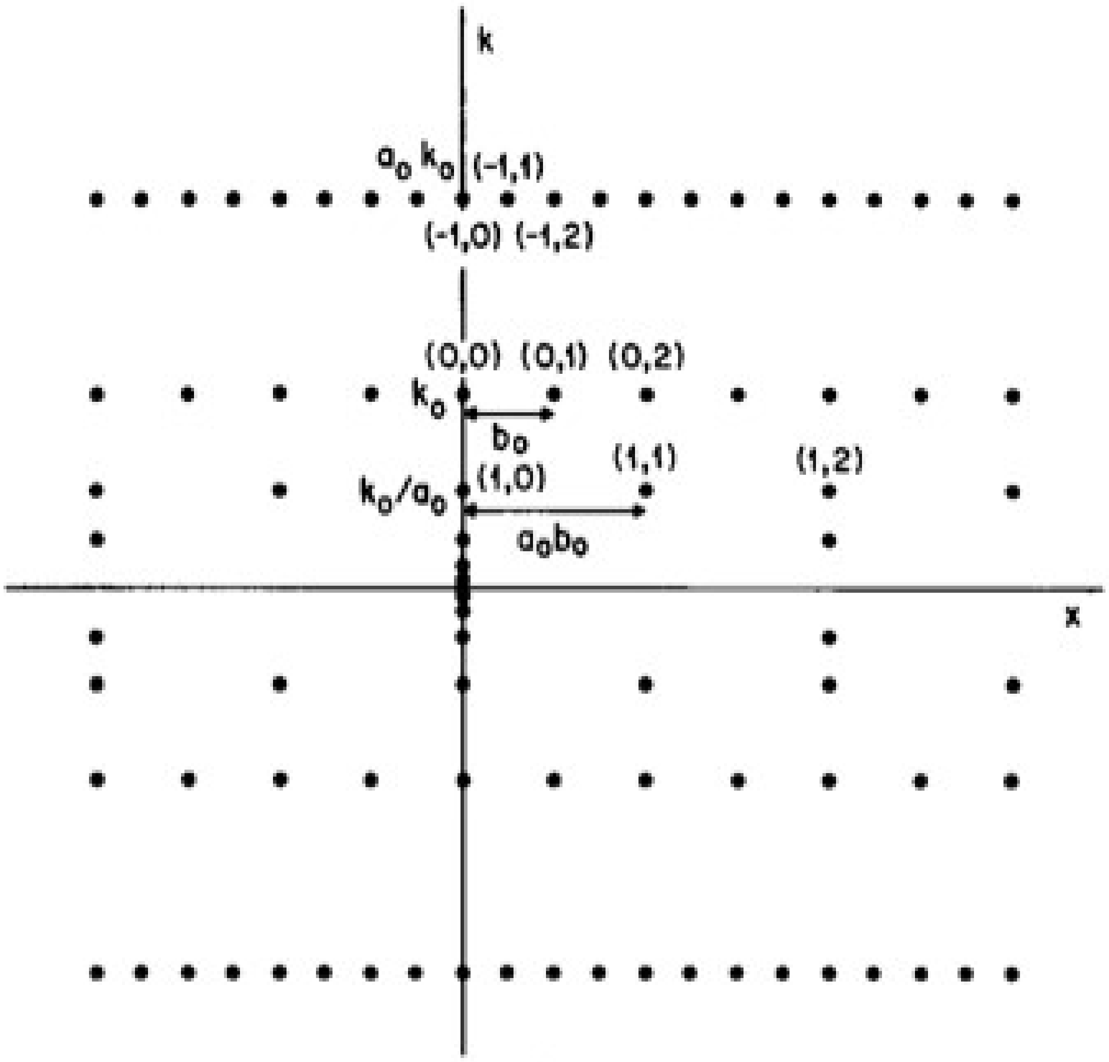
☆ Discrete Wavelet Transforms

How to sample $Wf(a,b)$??

⇒ another great insight by J. Morlet

"regular hyperbolic grid"

$$(a, b) = (a_0^m, n a_0^m b_0), \quad m, n \in \mathbb{Z}$$



a bit technical

Thm (Regular sampling thm, Daubechies '90)

Let ψ be a real-valued L^2-function.
For fixed a_0, b_0, define

$$\psi_{m,n}(x) := a_0^{-m/2}\,\psi(a_0^{-m}x - nb_0), \quad m, n \in \mathbb{Z}$$
$$= \frac{1}{\sqrt{a_0^m}}\,\psi\left(\frac{x - na_0^m b_0}{a_0^m}\right)$$

(1) If $\{\psi_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ is a frame of $L^2(\mathbb{R})$ with the frame bounds A, B, then we must have

$$A \leq \frac{1}{b_0}\sum_{-\infty}^{\infty} |\hat{\psi}(a_0^m \xi)|^2 \leq B \text{ for } \xi \in \mathbb{R} \text{ a.e.}$$

In particular, ψ satisfies the admissibility cond.

$$C_\psi = \int_0^\infty |\hat{\psi}(\xi)|^2 \frac{d\xi}{\xi} < +\infty$$

(2) If, for some $\varepsilon > 0$, ψ satisfies

$$|x|^{\frac{1}{2}+\varepsilon}\psi \in L^2, \quad |\xi|^\varepsilon \hat{\psi} \in L^2 \text{ and } \int \psi(x)\,dx = 0,$$

then ψ satisfies:

$$(*)\begin{cases} \operatorname{ess\,inf} \sum_{m\in\mathbb{Z}} |\hat{\psi}(a_0^m \xi)|^2 > 0 \\ \operatorname{ess\,sup} \sum_{m\in\mathbb{Z}} |\hat{\psi}(a_0^m \xi)|^2 < +\infty \end{cases} \text{ for any } a_0 \text{ close enough to } 1.$$

(i.e., $\exists\, \alpha = \alpha(\psi) > 1$ s.t. $(*)$ is satisfied $\forall a_0 \in (1, \alpha)$.)

Moreover, if b_0 is close enough to 0 (i.e., $\exists\, \beta = \beta(a_0, \psi)$ s.t. $(*)$ is satisfied $\forall\, b_0 \in (0, \beta)$),
then $\{\psi_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ constitute a frame!

Ex. $\psi(x)$ = the Mexican hat fcn
$a_0 = 2$, $b_0 = 1/4$.
$\Rightarrow \{\psi_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ forms a frame
(called a wavelet frame)
$A = 13.09$, $B = 14.18$ i.e., almost tight!

Dual Frame: Wavelet frame operator U commutes with dilations $\delta_{a_0^m}$,
but not with translations $\tau_{na_0^m b_0}$
$\Rightarrow$ dual frame $\{(U^*U)^{-1}\tau_{na_0^m b_0}\delta_{a_0^m}\psi\}_{(m,n)\in\mathbb{Z}^2}$
is in general not a wavelet system
(unlike the Gabor frame).