Lecture 16 : Wavelet Bases II

Note Title

✯ Conjugate Mirror Filters

A whole MRA is entirely characterized by the scaling fcn ϕ since it generates V_0 and consequently all V_j's. $j \in \mathbb{Z}$.

An interesting thing is that any scaling fcn is specified by a discrete filter called conjugate mirror filter (CMF).

- The scaling (or two-scale difference) eqn:

Recall $V_1 \subset V_0$, and $\frac{1}{\sqrt{2}}\phi\left(\frac{x}{2}\right) \in V_1$.

Hence $\frac{1}{\sqrt{2}}\phi\left(\frac{x}{2}\right) = \sum_{k\in\mathbb{Z}} h_k \, \phi(x-k)$

This is an expansion of $\frac{1}{\sqrt{2}}\phi\left(\frac{x}{2}\right) \in V_1 \subset V_0$ w.r.t. $\{\phi(x-k)\}_{k\in\mathbb{Z}}$ (an ONB of V_0).

$\mathcal{F}$

$$h_k = \left\langle \frac{1}{\sqrt{2}}\phi\left(\frac{\bullet}{2}\right), \phi(\bullet - k) \right\rangle$$

$$\sqrt{2}\,\hat{\phi}(2\xi) = \hat{h}(\xi)\,\hat{\phi}(\xi), \quad \hat{h}(\xi) := \sum_{k\in\mathbb{Z}} h_k e^{-2\pi i k\xi}$$

$$\Rightarrow \hat{\phi}(2\xi) = \frac{1}{\sqrt{2}}\hat{h}(\xi)\,\hat{\phi}(\xi)$$

i.e., $\hat{\phi}(\xi) = \frac{1}{\sqrt{2}}\hat{h}\left(\frac{\xi}{2}\right)\hat{\phi}\left(\frac{\xi}{2}\right)$

$$= \frac{1}{\sqrt{2}}\hat{h}\left(\frac{\xi}{2}\right)\hat{\phi}\left(\frac{\xi}{2^2}\right)$$

$$= \cdots$$

$$\Rightarrow \hat{\phi}(\xi) = \prod_{p=1}^{P}\left(\frac{\hat{h}(2^{-p}\xi)}{\sqrt{2}}\right)\hat{\phi}(2^{-P}\xi)$$

If $\hat{\phi}(\xi)$ is continuous at $\xi = 0$,
then $\hat{\phi}(2^{-P}\xi) \to \hat{\phi}(0)$ as $P \to \infty$

$$\Rightarrow \hat{\phi}(\xi) = \hat{\phi}(0) \prod_{p=1}^{\infty} \left(\frac{\hat{h}(2^{-p}\xi)}{\sqrt{2}} \right)$$

Thm (Mallat & Meyer, 1986?)

Let $\phi \in L^2(\mathbb{R})$ be an integrable scaling fcn, i.e., $\int \phi(x)\,dx < \infty$, $\{\phi(x-k)\}_{k\in\mathbb{Z}}$: an ONB of V_0.

necessary cond. $\Rightarrow |\hat{h}(\xi)|^2 + |\hat{h}(\xi+\frac{1}{2})|^2 \equiv 2$ a.e. $\xi \in \mathbb{R}$.

and $\hat{h}(0) = \sqrt{2}$.

Conversely, if $\hat{h}(\xi)$ satisfies:

sufficient cond.

1) 1-periodic;
2) C^1 in the neighborhood of $\xi = 0$; and
3) $|\hat{h}(\xi)|^2 + |\hat{h}(\xi+\frac{1}{2})|^2 \equiv 2$, $\hat{h}(0) = \sqrt{2}$,
$\inf_{\xi\in[-\frac{1}{4},\frac{1}{4}]} |\hat{h}(\xi)| > 0$,

then $\hat{\phi}(\xi) = \prod_{p=1}^{\infty} \frac{\hat{h}(2^{-p}\xi)}{\sqrt{2}}$ is the Fourier transform of a scaling fcn $\phi \in L^2(\mathbb{R})$.

(Proof) Here, we only prove the necessary cond. for the whole proof, see, e.g., Mallat's book.

$\{\phi(x-k)\}_{k\in\mathbb{Z}}$: an ONB for $V_0 \subset L^2(\mathbb{R})$.

The F.T. of the orthonormality gives as

$$(*) \quad \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi+k)|^2 \equiv 1 \quad \text{a.e. } \xi \in \mathbb{R}$$ as we did before.

By the two-scale diff. eqn. in the Fourier dom.

$\hat{\phi}(\xi) = \frac{1}{\sqrt{2}} \hat{h}\left(\frac{\xi}{2}\right) \hat{\phi}\left(\frac{\xi}{2}\right)$, (*) becomes

$$\sum_{k\in\mathbb{Z}} \left|\hat{h}\left(\frac{\xi}{2}+\frac{k}{2}\right)\right|^2 \left|\hat{\phi}\left(\frac{\xi}{2}+\frac{k}{2}\right)\right|^2 \equiv 2$$

$$\Longleftrightarrow \sum_{\ell\in\mathbb{Z}} \left|\hat{h}\left(\frac{\xi}{2}+\ell\right)\right|^2 \left|\hat{\phi}\left(\frac{\xi}{2}+\ell\right)\right|^2 + \left|\hat{h}\left(\frac{\xi}{2}+\frac{1}{2}+\ell\right)\right|^2 \left|\hat{\phi}\left(\frac{\xi}{2}+\frac{1}{2}+\ell\right)\right|^2 \equiv 2$$

$\hat{h}$: 1-periodic

$$\Longleftrightarrow \left|\hat{h}\left(\frac{\xi}{2}\right)\right|^2 \underbrace{\sum_{\ell\in\mathbb{Z}} \left|\hat{\phi}\left(\frac{\xi}{2}+\ell\right)\right|^2}_{1 \text{ via } (*)} + \left|\hat{h}\left(\frac{\xi}{2}+\frac{1}{2}\right)\right|^2 \underbrace{\sum_{\ell\in\mathbb{Z}} \left|\hat{\phi}\left(\frac{\xi}{2}+\frac{1}{2}+\ell\right)\right|^2}_{1 \text{ via } (*)} \equiv 2$$

$$\Longleftrightarrow \left|\hat{h}\left(\frac{\xi}{2}\right)\right|^2 + \left|\hat{h}\left(\frac{\xi}{2}+\frac{1}{2}\right)\right|^2 \equiv 2, \quad \text{a.e. } \xi\in\mathbb{R}$$

$$\Longleftrightarrow \left|\hat{h}(\xi)\right|^2 + \left|\hat{h}\left(\xi+\frac{1}{2}\right)\right|^2 \equiv 2, \quad \text{a.e. } \xi\in\mathbb{R} \checkmark$$

Now, $\hat{\phi}(2\xi) = \frac{1}{\sqrt{2}} \hat{h}(\xi)\hat{\phi}(\xi)$ and set $\xi = 0$

$$\Rightarrow \hat{\phi}(0) = \frac{1}{\sqrt{2}}\hat{h}(0)\hat{\phi}(0) \Longleftrightarrow \hat{h}(0) = \sqrt{2} \text{ since } \hat{\phi}(0) \neq 0.$$

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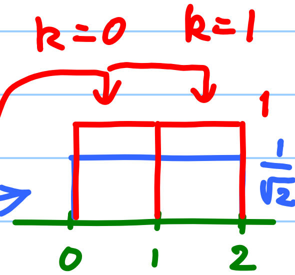
<u>Example 1</u> : Piecewise Const. MRA

$$\phi(x) = \chi_{[0,1)}(x)$$

$$h_k = \left\langle \underbrace{\frac{1}{\sqrt{2}}\chi_{[0,2)}}, \underbrace{\chi_{[0,1)}(\cdot - k)} \right\rangle$$



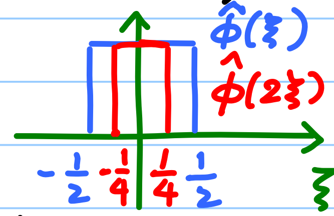
$$= \begin{cases} \frac{1}{\sqrt{2}} & k = 0, 1 \\ 0 & \text{o.w.} \quad (\text{no overlap}) \end{cases}$$

Example 2 : Shannon MRA

$$\phi(x) = \operatorname{sinc}(x), \quad \hat{\phi}(\xi) = \chi_{[-\frac{1}{2},\frac{1}{2})}(\xi)$$

From the two-scale diff. eqn. in Fourier,

$$\hat{h}(\xi) = \frac{\sqrt{2}\,\hat{\phi}(2\xi)}{\hat{\phi}(\xi)} = \sqrt{2}\,\chi_{[-\frac{1}{4},\frac{1}{4})}(\xi) \quad \text{for } \forall \xi \in [-\tfrac{1}{2},\tfrac{1}{2})$$



$$\hat{h}(\xi) = \sum_{k\in\mathbb{Z}} h_k e^{-2\pi i k\xi} = \sqrt{2}\,\chi_{[-\frac{1}{4},\frac{1}{4})}(\xi)$$

$$h_k = \int_{-1/2}^{1/2} \sqrt{2}\,\chi_{[-\frac{1}{4},\frac{1}{4})}(\xi)\, e^{+2\pi i k\xi}\, d\xi$$

$$= \sqrt{2}\int_{-1/4}^{1/4} e^{2\pi i k\xi}\, d\xi = \frac{1}{\sqrt{2}} \frac{\sin\frac{\pi k}{2}}{\frac{\pi k}{2}}$$

$$= \frac{1}{\sqrt{2}} \operatorname{sinc}\left(\frac{k}{2}\right), \quad k \in \mathbb{Z}.$$

$\Rightarrow \{h_k\}$: not a finite sequence.

Example 3 : Spline MRA

Recall

$$\hat{\phi}(\xi) = \frac{e^{-i\varepsilon\pi\xi}}{\xi^{m+1}\sqrt{S_{2m+2}(\xi)}} \qquad S_n(\xi) := \sum_{k\in\mathbb{Z}} (\xi+k)^{-n}$$

$$\hat{h}(\xi) = \frac{\sqrt{2}\,\hat{\phi}(2\xi)}{\hat{\phi}(\xi)} = e^{-i\varepsilon\pi\xi}\sqrt{\frac{S_{2m+2}(\xi)}{2^{2m+1} S_{2m+2}(2\xi)}}$$

$\underline{m=1}$: linear case $\Rightarrow$

Recall $S_4(\xi) = \dfrac{\pi^4}{3}\, \dfrac{1+2\cos^2\pi\xi}{\sin^4\pi\xi}$.

$$\Rightarrow \quad \hat{h}(\xi) = \sqrt{\frac{1}{2^3}\cdot\frac{1+2\cos^2\pi\xi}{1+2\cos^2 2\pi\xi}\cdot\frac{\sin^4 2\pi\xi}{\sin^4\pi\xi}}$$

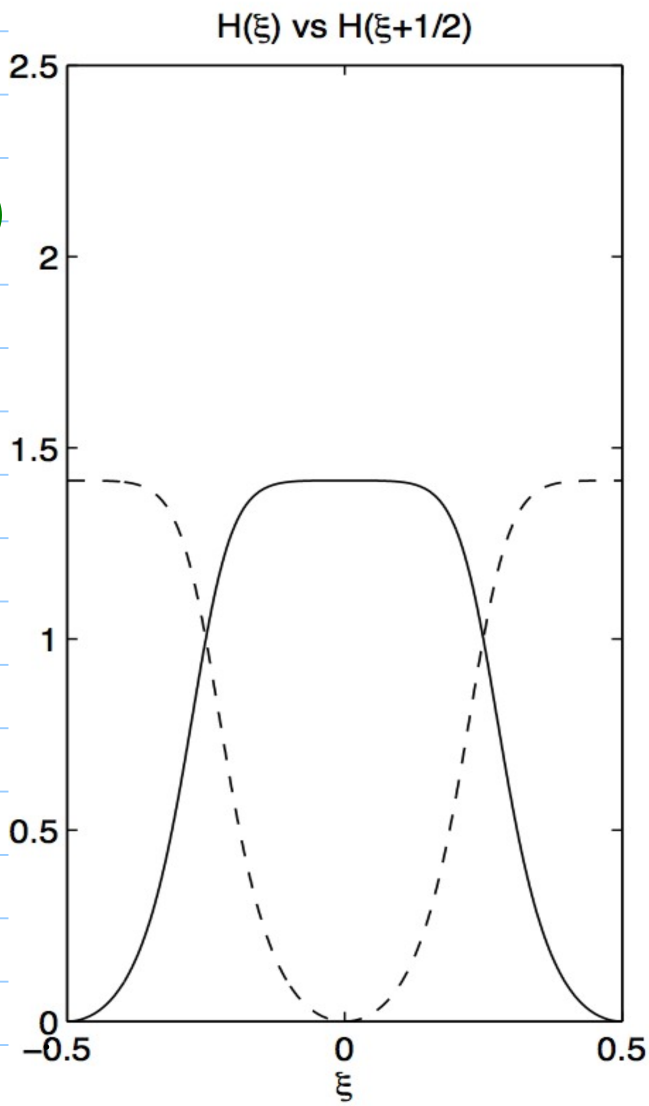
$$= \left(\frac{2\sin\pi\xi\cdot\cos\pi\xi}{\sin\pi\xi}\right)^4$$

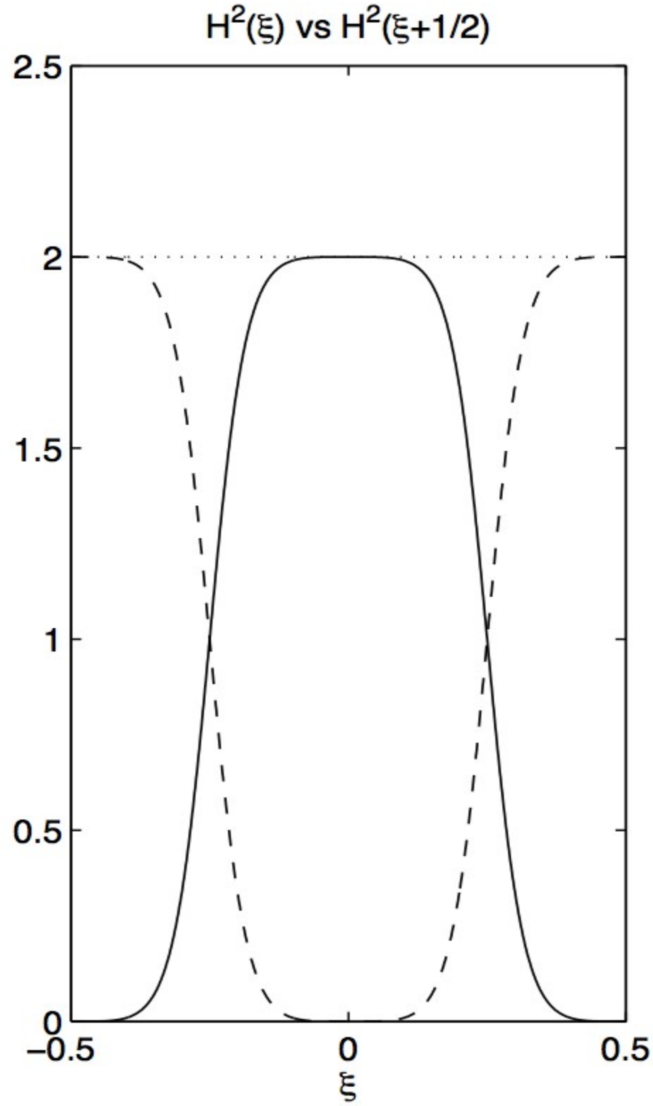
$$= \sqrt{2}\,\sqrt{\frac{1+2\cos^2\pi\xi}{1+2\cos^2 2\pi\xi}}\cdot\cos^2\pi\xi$$

$\Rightarrow \{h_k\}$: numerical table

spline scaling fcn may be relatively localized in x but not compactly supported while $\theta(x)$ is compactly supported.

$H(\xi) = \hat{h}(\xi)$





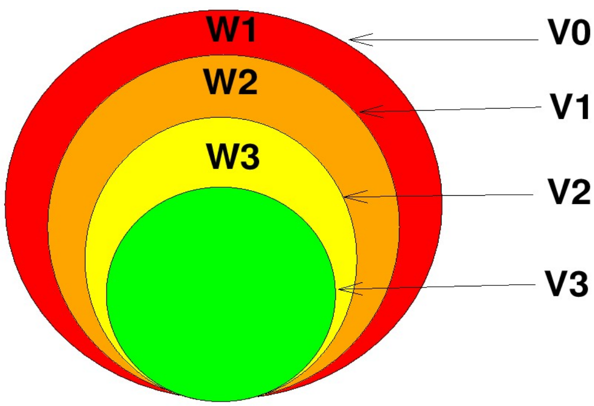
☆ Mother Wavelet; Wavelet ONB

Recall an MRA of $L^2(\mathbb{R})$

$$\cdots \subset V_{j+1} \subset V_j \subset V_{j-1} \subset \cdots$$

Consider the orthogonal complement of V_j in V_{j-1}, i.e., the information contained in V_{j-1} but not in V_j.

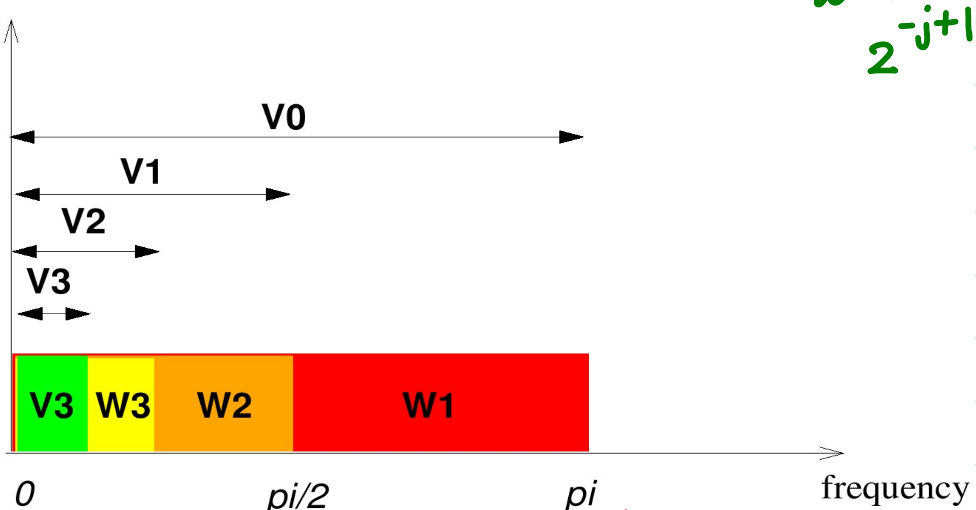
$$V_j \oplus W_j = V_{j-1}$$



The Concept of Multiresolution Analysis

In terms of the orthogonal proj.'s, we can write $\forall f \in L^2(\mathbb{R})$,

$$\underbrace{P_{V_{j-1}} f}_{\substack{\text{approx.} \\ \text{at resol.} \\ 2^{-j+1}}} = \underbrace{P_{V_j} f}_{\substack{\text{approx.} \\ \text{at resol.} \\ 2^{-j}}} + \underbrace{P_{W_j} f}_{\substack{\text{detailed info} \\ \text{necessary} \\ \text{to recover} \\ P_{V_{j-1}} f}}$$



Shannon

Multiresolution Analysis by Sinc Wavelets

Multiresolution Decomposition with Haar Basis

In fact, in this case → f

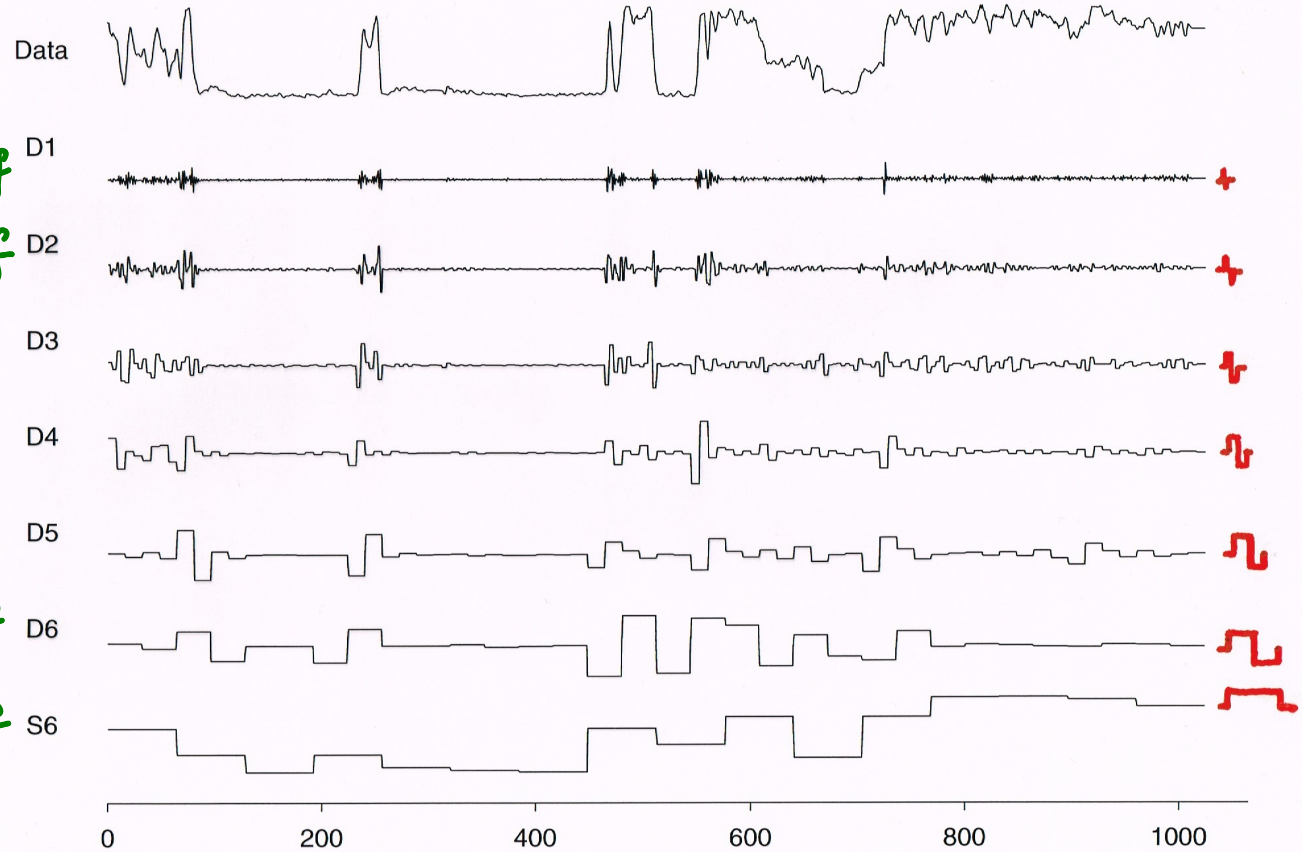
$f = P_{V_0} f$ $\quad P_{W_1} f$

$P_{W_2} f$

$\vdots$

$P_{W_6} f$

$P_{V_6} f$



Multiresolution Approximation with Haar Basis

f

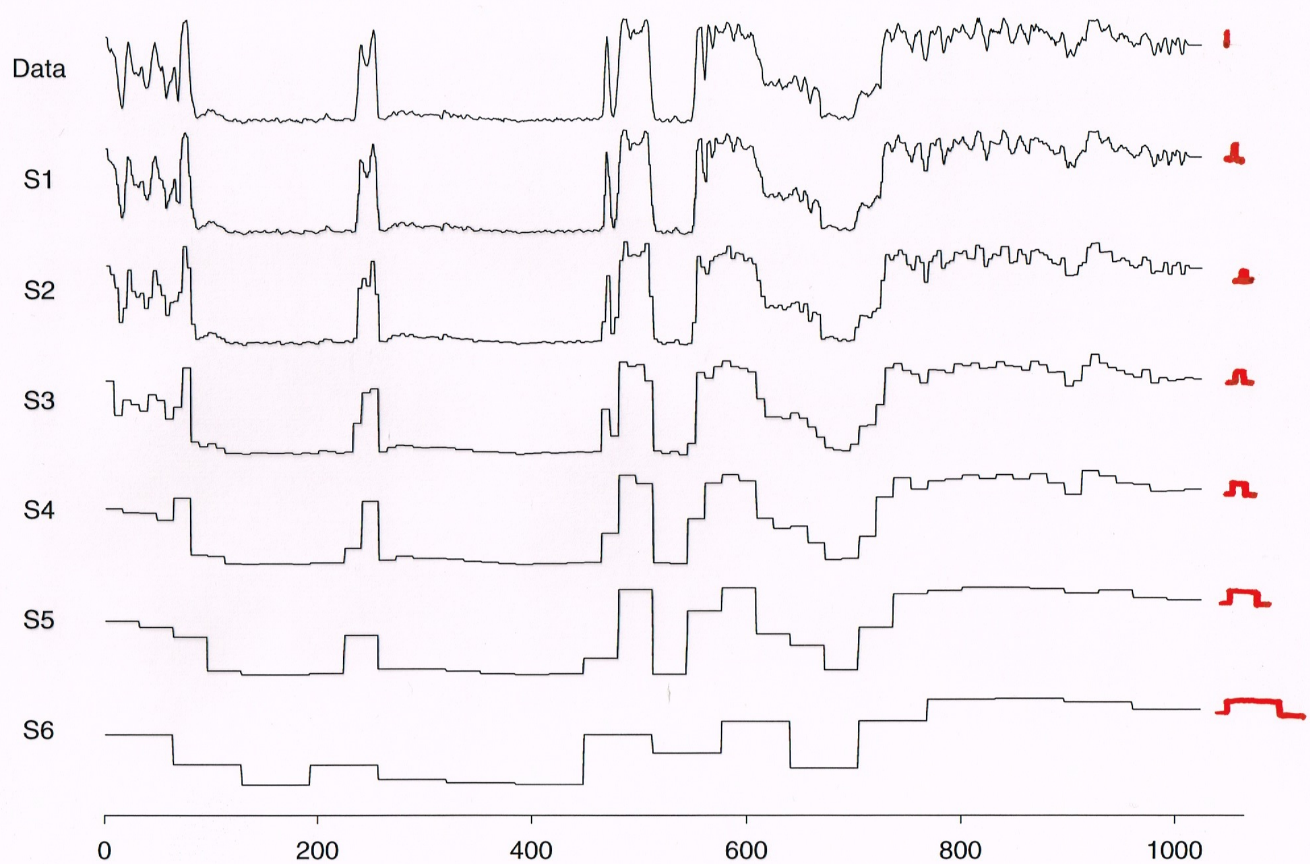
$P_{V_1} f$

$P_{V_2} f$

$\vdots$

$P_{V_6} f + P_{W_6} f \Rightarrow P_{V_5} f$

$P_{V_6} f$



Father $\phi \to \phi_{j,k}$, $V_j = \overline{\mathrm{span}\{\phi_{j,k}\}_{k\in\mathbb{Z}}}$ ONB
Mother $\psi \to \psi_{j,k}$, $W_j = \overline{\mathrm{span}\{\psi_{j,k}\}}_{k\in\mathbb{Z}}$ ONB

Thm (Mallat, Meyer 1986)
Let ϕ be a scaling fcn (father wavelet) and $\{h_k\}_{k\in\mathbb{Z}}$ be the corresponding CMF. Let us define $\psi \in L^2(\mathbb{R})$ whose Fourier transf. has

$$\hat{\psi}(\xi) = \frac{1}{\sqrt{2}}\hat{g}\left(\frac{\xi}{2}\right)\hat{\phi}\left(\frac{\xi}{2}\right)$$

with $\hat{g}(\xi) = e^{-2\pi i\xi}\,\overline{\hat{h}\left(\xi+\frac{1}{2}\right)}$.

Let $\psi_{j,k}(x) = 2^{-j/2}\psi(2^{-j}x-k)$.

Then, $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ form an ONB of W_j for each $j\in\mathbb{Z}$, and $\{\psi_{j,k}\}_{(j,k)\in\mathbb{Z}^2}$ form an ONB of $L^2(\mathbb{R})$.

(Proof) We look for a fcn $\psi \in L^2(\mathbb{R})$ s.t.

$$\frac{1}{\sqrt{2}}\psi\left(\frac{x}{2}\right) = \psi_{1,0}(x) \in W_1 \subset V_0$$

and $\{\psi_{1,k}\}_{k\in\mathbb{Z}}$ form an ONB of W_1.

Suppose $\frac{1}{\sqrt{2}}\psi\left(\frac{x}{2}\right) \in W_1$. Since $W_1 \subset V_0$ and $\{\phi(x-k)\}_{k\in\mathbb{Z}}$: an ONB of V_0,

$$\frac{1}{\sqrt{2}}\psi\left(\frac{x}{2}\right) = \sum_{k\in\mathbb{Z}} \underbrace{\left\langle \frac{1}{\sqrt{2}}\psi\left(\frac{\cdot}{2}\right), \phi(\cdot-k)\right\rangle}_{=:\,g_k} \phi(x-k)$$

$$\Big\downarrow \mathcal{F}$$

$$\sqrt{2}\,\hat{\psi}(2\xi) = \hat{g}(\xi)\hat{\phi}(\xi), \quad \hat{g}(\xi) = \sum_{k\in\mathbb{Z}} g_k e^{-2\pi i k\xi}.$$

Lemma The family $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ is an ONB of W_j

$$\iff \begin{cases} |\hat{g}(\xi)|^2 + |\hat{g}(\xi+\frac{1}{2})|^2 \equiv 2 & \text{a.e. } \xi\in\mathbb{R} \\ \hat{g}(\xi)\overline{\hat{h}(\xi)} + \hat{g}(\xi+\frac{1}{2})\overline{\hat{h}(\xi+\frac{1}{2})} \equiv 0 \end{cases}$$

(Proof of Lemma) We'll prove only $j=0$ case since the other cases are easy via δ_{2^j} op. once we prove the $j=0$ case.

Using the same argument in the proof of $\{\phi(x-k)\}_{k\in\mathbb{Z}}$ forming an ONB of V_0, we can show that

$\{\psi(x-k)\}_{k\in\mathbb{Z}}$ are orthonormal

$$\iff I(\xi) := \sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi+k)|^2 \equiv 1 \quad \text{a.e. } \xi\in\mathbb{R}$$

Now, the two-scale diff. eqn. $\hat{\psi}(\xi) = \frac{1}{\sqrt{2}}\hat{g}(\frac{\xi}{2})\hat{\phi}(\frac{\xi}{2})$

$$I(\xi) = \frac{1}{2}\sum_k |\hat{g}(\tfrac{\xi}{2}+\tfrac{k}{2})|^2 |\hat{\phi}(\tfrac{\xi}{2}+\tfrac{k}{2})|^2 \qquad \hat{g}: \text{1-periodic}$$

$$= \frac{1}{2}\sum_\ell \Big(|\hat{g}(\tfrac{\xi}{2}+\ell)|^2 |\hat{\phi}(\tfrac{\xi}{2}+\ell)|^2 + |\hat{g}(\tfrac{\xi}{2}+\ell+\tfrac{1}{2})|^2 |\hat{\phi}(\tfrac{\xi}{2}+\ell+\tfrac{1}{2})|^2 \Big)$$

$\hat{g}$: 1-periodic

$$\overset{\downarrow}{=} \frac{1}{2}\Big\{ |\hat{g}(\tfrac{\xi}{2})|^2 \underbrace{\sum_\ell |\hat{\phi}(\tfrac{\xi}{2}+\ell)|^2}_{=1} + |\hat{g}(\tfrac{\xi}{2}+\tfrac{1}{2})|^2 \underbrace{\sum_\ell |\hat{\phi}(\tfrac{\xi}{2}+\ell+\tfrac{1}{2})|^2} \Big\}$$

$$= \frac{1}{2}\Big(|\hat{g}(\tfrac{\xi}{2})|^2 + |\hat{g}(\tfrac{\xi}{2}+\tfrac{1}{2})|^2 \Big) \equiv 1 \quad \text{a.e. } \xi\in\mathbb{R}$$

$$\iff |\hat{g}(\xi)|^2 + |\hat{g}(\xi+\tfrac{1}{2})|^2 \equiv 2 \quad \text{a.e. } \xi\in\mathbb{R}.$$

Now, $W_0 \perp V_0 \iff \{\phi(x-k)\}_{k\in\mathbb{Z}} \perp \{\psi(x-k)\}_{k\in\mathbb{Z}}$

Let's check whether $\psi(x) \perp \phi(x-k)$.

$$\langle \psi(\cdot), \phi(\cdot - k)\rangle = \int \psi(x)\overline{\phi(x-k)}\,dx$$

$$= (\psi * \tilde{\phi})(k) \overset{?}{=} 0$$

$$\begin{tikzcd}
\psi * \tilde{\phi}(x) \arrow[r, "\mathcal{F}"] \arrow[d, "\substack{\text{sampling}\\ \text{at } x=k}"'] & \hat{\psi}(\xi)\overline{\hat{\phi}(\xi)} \arrow[d, "\text{periodization}"] \\
\psi * \tilde{\phi}(k) & \sum_{\ell} \hat{\psi}(\xi+\ell)\overline{\hat{\phi}(\xi+\ell)}
\end{tikzcd}$$

$$\hat{\psi}(\xi) = \frac{1}{\sqrt{2}}\hat{g}\left(\frac{\xi}{2}\right)\hat{\phi}\left(\frac{\xi}{2}\right), \quad \hat{\phi}(\xi) = \frac{1}{\sqrt{2}}\hat{h}\left(\frac{\xi}{2}\right)\hat{\phi}\left(\frac{\xi}{2}\right).$$

$$\Rightarrow \sum_{\ell} \hat{\psi}(\xi+\ell)\overline{\hat{\phi}(\xi+\ell)}$$

$$= \frac{1}{2}\sum_{\ell} \hat{g}\left(\frac{\xi+\ell}{2}\right)\overline{\hat{h}\left(\frac{\xi+\ell}{2}\right)}\left|\hat{\phi}\left(\frac{\xi+\ell}{2}\right)\right|^2$$

$$= \frac{1}{2}\sum_{m} \hat{g}\left(\frac{\xi}{2}+m\right)\overline{\hat{h}\left(\frac{\xi}{2}+m\right)}\left|\hat{\phi}\left(\frac{\xi}{2}+m\right)\right|^2$$

$\hat{h}, \hat{g}$: 1-periodic

$$+ \hat{g}\left(\frac{\xi}{2}+m+\frac{1}{2}\right)\overline{\hat{h}\left(\frac{\xi}{2}+m+\frac{1}{2}\right)}\left|\hat{\phi}\left(\frac{\xi}{2}+m+\frac{1}{2}\right)\right|^2$$

$$= \frac{1}{2}\left(\hat{g}\left(\frac{\xi}{2}\right)\overline{\hat{h}\left(\frac{\xi}{2}\right)} + \hat{g}\left(\frac{\xi}{2}+\frac{1}{2}\right)\overline{\hat{h}\left(\frac{\xi}{2}+\frac{1}{2}\right)}\right) \cdot \underbrace{\sum_{\ell}\left|\hat{\phi}\left(\frac{\xi}{2}+\frac{\ell}{2}\right)\right|^2}_{=1}$$

Hence, $\psi * \tilde{\phi}(k) = 0$

$$\iff \hat{g}(\xi)\overline{\hat{h}(\xi)} + \hat{g}\left(\xi+\frac{1}{2}\right)\overline{\hat{h}\left(\xi+\frac{1}{2}\right)} \equiv 0$$

a.e. $\xi \in \mathbb{R}$.

Finally, we need to show $V_{-1} = V_0 \oplus W_0$.
We know $\{\sqrt{2}\,\phi(2x-k)\}_{k\in\mathbb{Z}}$ form an ONB of V_{-1}.

So, $V_{-1} = V_0 \oplus W_0$

$\iff$ $\forall\{a_k\} \in \ell^2(\mathbb{Z})$, $\exists\{b_k\}, \{c_k\} \in \ell^2(\mathbb{Z})$ s.t.

$$\sum a_k \sqrt{2}\,\phi\left(2\left(x-\frac{k}{2}\right)\right) = \sum b_k \phi(x-k) + \sum c_k \psi(x-k)$$

$$\Big\downarrow \mathcal{F}$$

$$\frac{1}{\sqrt{2}}\hat{a}\left(\frac{\xi}{2}\right)\hat{\phi}\left(\frac{\xi}{2}\right) = \hat{b}(\xi)\hat{\phi}(\xi) + \hat{c}(\xi)\hat{\psi}(\xi)$$

$$\iff \hat{a}\left(\frac{\xi}{2}\right) = \hat{b}(\xi)\hat{h}\left(\frac{\xi}{2}\right) + \hat{c}(\xi)\hat{g}\left(\frac{\xi}{2}\right) \quad \text{—} (*)$$

via $\hat{\phi}(\xi) = \frac{1}{\sqrt{2}}\hat{h}\left(\frac{\xi}{2}\right)\hat{\phi}\left(\frac{\xi}{2}\right)$, $\hat{\psi}(\xi) = \frac{1}{\sqrt{2}}\hat{g}\left(\frac{\xi}{2}\right)\hat{\phi}\left(\frac{\xi}{2}\right)$

Question: Do such $\{b_k\}, \{c_k\}$ exist?
$\Rightarrow$ Yes!

Define
$$\hat{b}(2\xi) := \frac{1}{2}\left[\hat{a}(\xi)\overline{\hat{h}(\xi)} + \hat{a}\left(\xi+\frac{1}{2}\right)\overline{\hat{h}\left(\xi+\frac{1}{2}\right)}\right]$$
$$\hat{c}(2\xi) := \frac{1}{2}\left[\hat{a}(\xi)\overline{\hat{g}(\xi)} + \hat{a}\left(\xi+\frac{1}{2}\right)\overline{\hat{g}\left(\xi+\frac{1}{2}\right)}\right].$$

Then these satisfy (*).
In fact,
$$\begin{cases} \hat{b}(\xi)\hat{h}\left(\frac{\xi}{2}\right) = \frac{1}{2}\left[\hat{a}\left(\frac{\xi}{2}\right)\left|\hat{h}\left(\frac{\xi}{2}\right)\right|^2 + \hat{a}\left(\frac{\xi}{2}+\frac{1}{2}\right)\hat{h}\left(\frac{\xi}{2}\right)\overline{\hat{h}\left(\frac{\xi}{2}+\frac{1}{2}\right)}\right] \\ \hat{c}(\xi)\hat{g}\left(\frac{\xi}{2}\right) = \frac{1}{2}\left[\hat{a}\left(\frac{\xi}{2}\right)\left|\hat{g}\left(\frac{\xi}{2}\right)\right|^2 + \hat{a}\left(\frac{\xi}{2}+\frac{1}{2}\right)\hat{g}\left(\frac{\xi}{2}\right)\overline{\hat{g}\left(\frac{\xi}{2}+\frac{1}{2}\right)}\right] \end{cases}$$

We can show that $\left|\hat{h}\left(\frac{\xi}{2}\right)\right|^2 + \left|\hat{g}\left(\frac{\xi}{2}\right)\right|^2 \equiv 2$ a.e. $\xi \in \mathbb{R}$
and $\hat{h}\left(\frac{\xi}{2}\right)\overline{\hat{h}\left(\frac{\xi}{2}+\frac{1}{2}\right)} + \hat{g}\left(\frac{\xi}{2}\right)\overline{g\left(\frac{\xi}{2}+\frac{1}{2}\right)} \equiv 0$

These can be derived from

$$\begin{cases} |\hat{h}(\tfrac{\xi}{2})|^2 + |\hat{h}(\tfrac{\xi}{2}+\tfrac{1}{2})|^2 \equiv 2 \\ |\hat{g}(\tfrac{\xi}{2})|^2 + |\hat{g}(\tfrac{\xi}{2}+\tfrac{1}{2})|^2 \equiv 2 \\ \hat{h}(\tfrac{\xi}{2})\overline{\hat{g}(\tfrac{\xi}{2})} + \hat{h}(\tfrac{\xi}{2}+\tfrac{1}{2})\overline{\hat{g}(\tfrac{\xi}{2}+\tfrac{1}{2})} \equiv 0 \end{cases} \quad \text{a.e. } \xi \in \mathbb{R}$$

Hence such $\hat{b}(\xi)$, $\hat{c}(\xi)$ exist.

They are 1-periodic because of their forms and $\hat{a}, \hat{h}, \hat{g}$ are also 1-periodic

Thus $\exists \{b_k\}, \{c_k\} \in \ell^2(\mathbb{Z})$

i.e., $V_{-1} = V_0 \oplus W_0$!

$\iff W_0 = V_0^{\perp}$ in V_{-1}

$\Rightarrow W_j = V_j^{\perp}$ in V_{j-1}, $V_{j-1} = V_j \oplus W_j$

Lemma done.

Now,

$W_j \perp V_j$, $W_\ell \subset V_{\ell-1} \subset V_j \quad \forall \ell > j$

$\Rightarrow W_j \perp W_\ell$. Hence $L^2(\mathbb{R}) = \bigoplus_{j=-\infty}^{\infty} W_j$

and for any $L > J$,

$$V_J = V_L \oplus W_L \oplus W_{L-1} \oplus \cdots \oplus W_{J-1}$$
$$= V_L \oplus \bigoplus_{j=L}^{J-1} W_j$$

Thm done.