

Lecture 17: Wavelet Bases III

Note Title

Recall

Thm (Mallat, Meyer 1986)

Let ϕ be a scaling fcn (father wavelet) and $\{h_k\}_{k \in \mathbb{Z}}$ be the corresponding CMF. Let us define $\psi \in L^2(\mathbb{R})$ whose Fourier transf. has

$$\hat{\psi}(\xi) = \frac{1}{\sqrt{2}} \hat{g}\left(\frac{\xi}{2}\right) \hat{\phi}\left(\frac{\xi}{2}\right)$$

with $\hat{g}(\xi) = e^{-2\pi i \xi} \overline{\hat{h}\left(\xi + \frac{1}{2}\right)}$.

Let $\psi_{j,k}(x) = 2^{-j/2} \psi(2^{-j}x - k)$.

Then, $\{\psi_{j,k}\}_{k \in \mathbb{Z}}$ form an ONB of W_j for each $j \in \mathbb{Z}$, and $\{\psi_{j,k}\}_{(j,k) \in \mathbb{Z}^2}$ form an ONB of $L^2(\mathbb{R})$.

Remarks:

(1) Let's check the relationship between $\{g_k\}_{k \in \mathbb{Z}}$ and $\{h_k\}_{k \in \mathbb{Z}}$.

But, first, recall

$$\left\{ \begin{aligned} h_k &= \langle \phi_{1,0}, \phi_{0,k} \rangle = \langle \phi(\cdot/2), \phi(\cdot - k) \rangle \\ g_k &= \langle \psi_{1,0}, \phi_{0,k} \rangle = \langle \psi(\cdot/2), \phi(\cdot - k) \rangle \end{aligned} \right.$$

from the two-scale diff. eqn's. and $V_1 \oplus W_1 = V_0$.

Now,

$$\hat{g}(\xi) = e^{-2\pi i \xi} \overline{\hat{h}\left(\xi + \frac{1}{2}\right)}$$

via \mathcal{F}^{-1}

This choice of \hat{g} satisfies all the cond's as \hat{g} .

$$g_k = (-1)^{1-k} h_{1-k}, \quad k \in \mathbb{Z}.$$

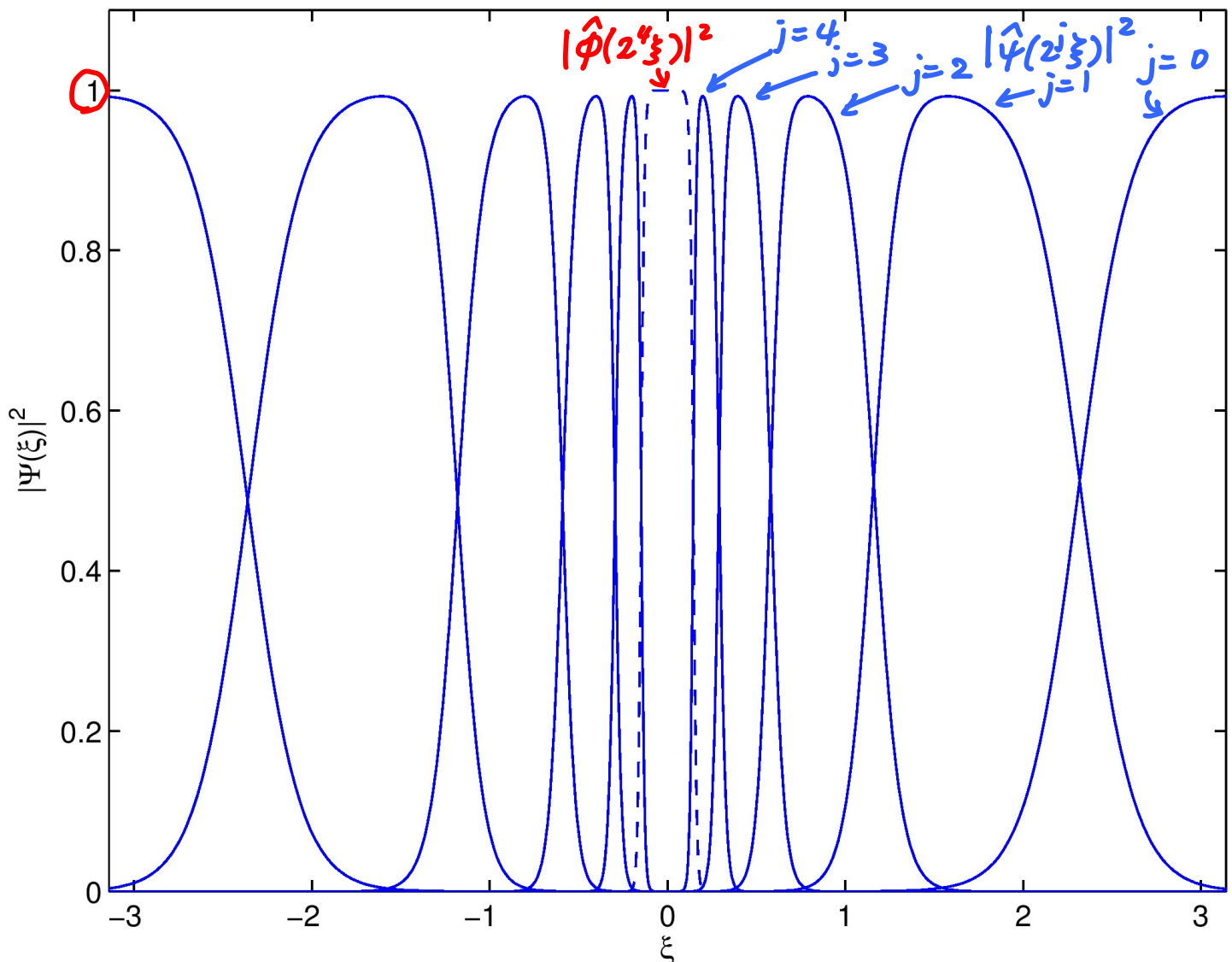
$$\begin{aligned}
 \hat{g}(\xi) &= e^{-2\pi i \xi} \overline{\hat{h}(\xi + \frac{1}{2})} \\
 &= e^{-2\pi i \xi} \sum h_l e^{+2\pi i l (\xi + \frac{1}{2})} \\
 &= e^{-2\pi i \xi} \sum h_l e^{+2\pi i l \xi} \cdot e^{\pi i l} = (-1)^l \\
 &= \sum (-1)^l h_l e^{2\pi i (l-1)\xi}
 \end{aligned}$$

$$1-l=k \quad \rightarrow \quad \sum_k (-1)^{1-k} h_{1-k} e^{-2\pi i k \xi}$$

$$= g_k$$

(2) $|\hat{\psi}(2^j \xi)|^2$ works as a **bandpass** filter.

also $\sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi)|^2 \equiv 1 \quad \text{a.e. } \xi \in \mathbb{R} \setminus \{0\}$



Example 1: Haar wavelet

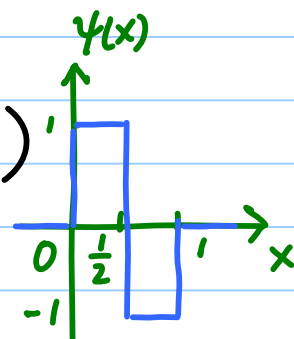
$$\phi(x) = \chi_{[0,1)}(x) \Rightarrow \psi(x) = \begin{cases} -1 & 0 < x < \frac{1}{2} \\ 1 & \frac{1}{2} < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

Recall $h_k = \begin{cases} \frac{1}{\sqrt{2}} & k=0,1 \\ 0 & \text{o.w.} \end{cases}$

$$\begin{aligned} \Rightarrow \frac{1}{\sqrt{2}} \psi\left(\frac{x}{2}\right) &= \sum (-1)^{1-k} h_{1-k} \phi(x-k) \\ &= -h_1 \phi(x) + h_0 \phi(x-1) \\ &= \frac{1}{\sqrt{2}} \left(-\chi_{[0,1)}(x) + \chi_{[1,2)}(x) \right) \end{aligned}$$

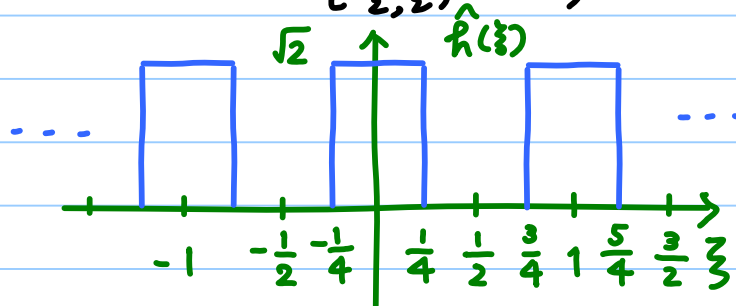
$$\Rightarrow \psi(x) = -\chi_{[0,\frac{1}{2})}(x) + \chi_{[\frac{1}{2},1)}(x)$$

$$= \frac{1}{\sqrt{2}} \left[\sqrt{2} \phi(2x-1) - \sqrt{2} \phi(2x) \right]$$



Example 2: Shannon wavelet

Recall: $\hat{\phi}(\xi) = \chi_{[-\frac{1}{2}, \frac{1}{2})}(\xi)$, $\hat{h}(\xi) = \sqrt{2} \chi_{[-\frac{1}{4}, \frac{1}{4})}(\xi)$



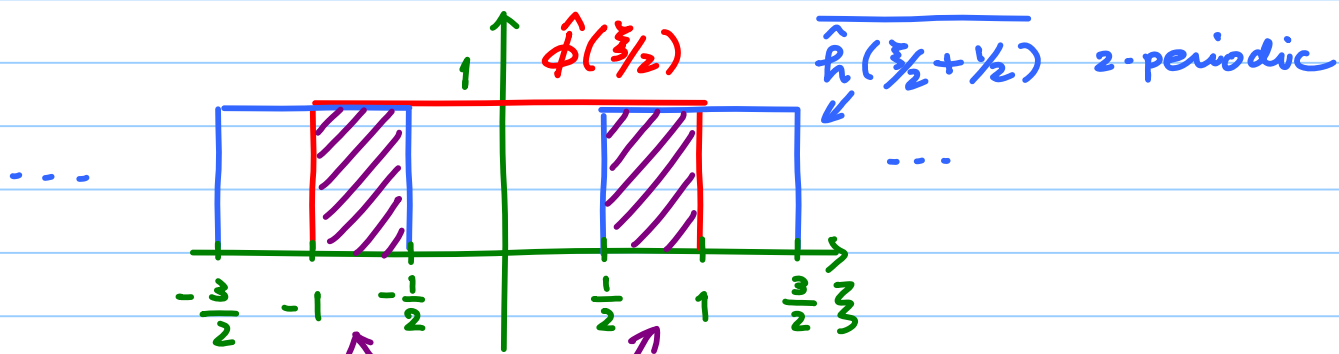
1-periodic
 $= \sqrt{2} \sum_k \chi_{[-\frac{1}{4}, \frac{1}{4})}(\xi+k)$

$$\hat{\psi}(\xi) = \frac{1}{\sqrt{2}} \hat{g}\left(\frac{\xi}{2}\right) \hat{\phi}\left(\frac{\xi}{2}\right) = \frac{1}{\sqrt{2}} e^{-i\pi\xi} \hat{h}\left(\frac{\xi}{2} + \frac{1}{2}\right) \hat{\phi}\left(\frac{\xi}{2}\right)$$

$$= e^{-i\pi\xi} \chi_{[-\frac{3}{2}, -\frac{1}{2})}(\xi) \chi_{[-1,1)}(\xi) \sqrt{2} \chi_{[-\frac{3}{2}, -\frac{1}{2})}(\xi)$$

2-periodic 2-periodic

$$= e^{-i\pi\xi} \left(\chi_{[-1,1)}(\xi) - \chi_{[-\frac{1}{2}, \frac{1}{2})}(\xi) \right)$$



$|\hat{\psi}(\xi)|$ or $e^{i\pi\xi} \hat{\psi}(\xi)$
 ideal (or pure) bandpass filter (could be too sharp in practice).

Hence via \mathcal{F}^{-1} , we have

$$\psi(x) = 2 \operatorname{sinc}(2x-1) - \operatorname{sinc}(x-\frac{1}{2})$$

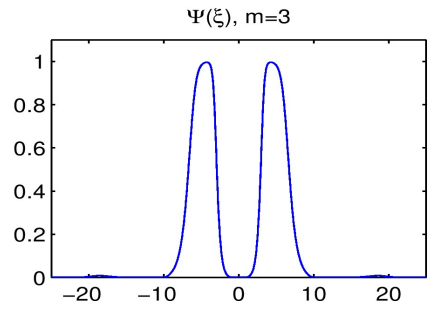
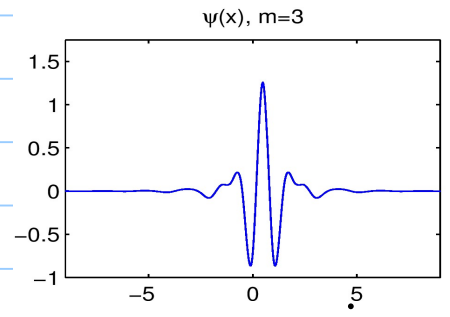
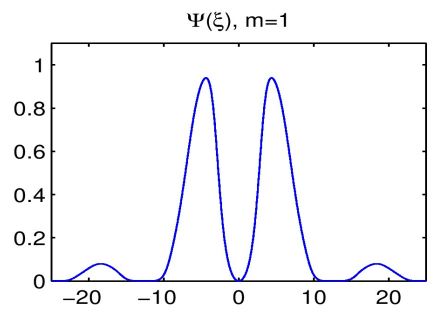
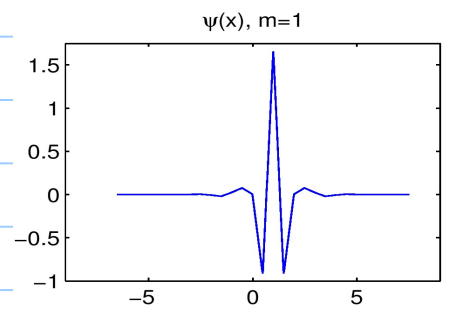
$$= \sqrt{2} \left[\sqrt{2} \phi(2x-1) - \frac{1}{\sqrt{2}} \tau_{\frac{1}{2}} \phi(x) \right]$$

Example 3: Spline (a.k.a. Battle-Lemarié) wavelet

$m=0$: the Haar wavelet

For general $m \in \mathbb{N}$

$$\hat{\psi}(\xi) = \frac{e^{-i\pi\xi}}{\xi^{m+1}} \sqrt{\frac{S_{2m+2}(\xi/2 + 1/2)}{S_{2m+2}(\xi) S_{2m+2}(\xi/2)}}$$



★ Basic Properties of Wavelets

⇒ Important in choosing/designing wavelets.

(1) Vanishing Moments

Def. A mother wavelet ψ is said to have

p **vanishing moments** if

$$\int_{-\infty}^{\infty} x^m \psi(x) dx = 0, \quad m = \underbrace{0, 1, \dots, p-1}$$

i.e., $\psi \perp P_{p-1}[x]$ (any polynomial with $\text{deg.} \leq p-1$).

If your fcn $f(x)$ is locally C^m , then f is well-approximated by a Taylor polynomial of $\text{deg } m$ over some short interval.

⇒ $\langle f, \psi_{j,k} \rangle \approx 0$ for small j if $m < p$.

So, in general, the larger p , the better.

Thm (Vanishing moments)

Let ϕ & ψ be father & mother wavelets generating a wavelet ONB for $L^2(\mathbb{R})$ with $|\phi(x)| = O((1+x^2)^{-p/2-1})$, $|\psi(x)| = O((1+x^2)^{-p/2-1})$.

Then the following statements are equivalent:

(a) ψ has p vanishing moments.

(b) $\hat{\psi}^{(m)}(0) = 0$, $m = 0, 1, \dots, p-1$.

(c) $\hat{h}^{(m)}(\frac{1}{2}) = 0$, $m = 0, 1, \dots, p-1$.

(d) $\forall 0 \leq m < p$,

$\xi_m(x) := \sum_{k \in \mathbb{Z}} k^m \phi(x-k)$ is a polynomial of $\text{deg. } m$.

(Proof) See Mallat's book Sec. 7.2. for the detail.

Note that

$$\hat{\psi}^{(m)}(\xi) = \mathcal{F} [(-2\pi i x)^m \psi(x)] \Rightarrow \hat{\psi}^{(m)}(0) = (-2\pi i)^m \int_{-\infty}^{\infty} x^m \psi(x) dx.$$

(2) Support Size

If your f has an isolated singularity at $x = x_0$ and if $x_0 \in \text{supp } \psi_{j,k}$, then $|\langle f, \psi_{j,k} \rangle|$ may be large.

\Rightarrow better to have $\text{supp } \psi$: **compact**!

Thm (Compact Support)

The father wavelet ϕ has a compact support $\Leftrightarrow \{h_k\}$ has a compact support (i.e., finite taps).

Their supports are equal.

a slight abuse of def. of supp.

If $\text{supp } \phi = \text{supp } h = [K_1, K_2]$, $K_1, K_2 \in \mathbb{Z}$, then $\text{supp } \psi = [(\frac{K_1 - K_2 + 1}{2}), (\frac{K_2 - K_1 + 1}{2})]$.

interval

list of indices i.e., $\{k \in \mathbb{Z} \mid h_k \neq 0\}$

(Proof) (\Rightarrow) Recall $h_k = \langle \phi_{1,0}, \phi_{0,k} \rangle$.

Since both $\phi_{1,0}$ & $\phi_{0,k}$ have compact supports, $\{h_k\}$ must be of finite taps.

(\Leftarrow) See Daubechies (1988, Comm. Pure & Appl. Math.)

Now, check the supports of ϕ and $\{h_k\}$.

Suppose $h_k \neq 0$ for $K_1 \leq k \leq K_2$, and $\text{supp } \phi = [N_1, N_2]$. Then, $\text{supp } \phi_{1,0} = [2N_1, 2N_2]$

On the other hand,

$$\phi_{1,0}(x) = \frac{1}{\sqrt{2}} \phi\left(\frac{x}{2}\right) = \sum_{k \in \mathbb{Z}} h_k \phi(x-k)$$

$$\Rightarrow \text{supp } \phi_{1,0} = [K_1 + N_1, K_2 + N_2]$$

$$\text{Hence } 2N_1 = K_1 + N_1, \quad 2N_2 = K_2 + N_2$$

$$\text{i.e., } N_1 = K_1, \quad N_2 = K_2. \quad \checkmark$$

As for $\text{supp } \psi$, recall

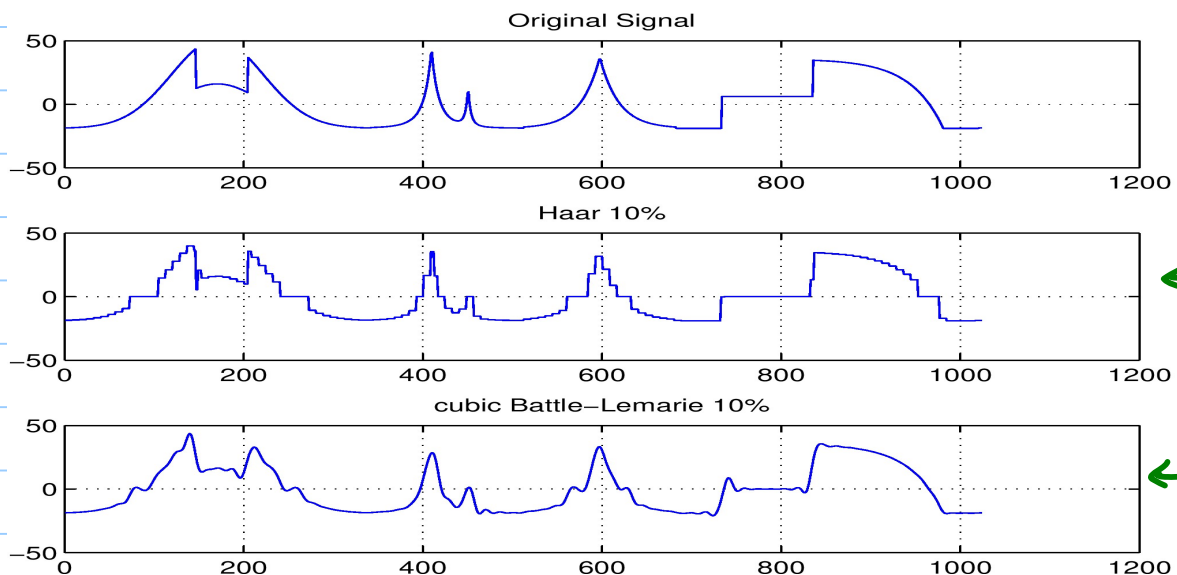
$$\psi_{1,0}(x) = \frac{1}{\sqrt{2}} \psi\left(\frac{x}{2}\right) = \sum_{k \in \mathbb{Z}} g_k \phi(x-k) = \sum_{k \in \mathbb{Z}} (-1)^{1-k} h_{1-k} \phi(x-k)$$

Since $\text{supp } \phi = \text{supp } h = [K_1, K_2]$, the support of the above two-scale diff. eqn. is $[K_1 - K_2 + 1, K_2 - K_1 + 1]$, $1 - k \in [K_1, K_2] \Leftrightarrow k \in [1 - K_2, 1 - K_1]$ which is $\text{supp } \psi_{1,0}$.
Hence, $\text{supp } \psi = [(K_1 - K_2 + 1)/2, (K_2 - K_1 + 1)/2]$ ///

Remark: The support size & the number of vanishing moments of ψ seems independent. However, Daubechies showed that for orthonormal wavelets, ψ has p vanishing moments $\Rightarrow |\text{supp } \psi| \geq 2p - 1$.
 \exists a **trade-off** between these two concepts. Daubechies's orthonormal wavelets achieve $|\text{supp } \psi| = 2p - 1$ with p vanishing moments.

(3) Regularity

When you reconstruct (or approximate) f from its compressed representation (e.g., from a finite subset of $\{ \langle f, \psi_{j,k} \rangle \}$, the regularity (i.e., smoothness) of $\psi_{j,k}$ matters.



Thm (Tchamitchian 1987)

Let $\hat{h}(\xi)$ be a CMF with p zeros at $\xi = \frac{1}{2}$ which satisfies the sufficient cond.'s of the Mallat - Meyer Thm, i.e.,

- 1) 1-periodic;
- 2) C^1 in the neighborhood of $\xi = 0$; and
- 3) $|\hat{h}(\xi)|^2 + |\hat{h}(\xi + \frac{1}{2})|^2 \equiv 2$; $\hat{h}(0) = \sqrt{2}$;
 $\inf_{\xi \in [-\frac{1}{4}, \frac{1}{4}]} |\hat{h}(\xi)| > 0$.

Now, let's perform the factorization:

$$\hat{h}(\xi) = \sqrt{2} \left(\frac{1 + e^{i2\pi\xi}}{2} \right)^p \hat{l}(\xi)$$

If $\sup_{\xi \in \mathbb{R}} |\hat{l}(\xi)| = B$, then ϕ & ψ are

uniformly Lipschitz (or Hölder) α for
 $\alpha < \alpha_0 := p - \log_2 B - 1$.

(Proof) See Mallat's book Sec. 7.2.

Note that we refine the definition of Lipschitz continuity in Lecture 5 as follows:

Def A fcn f is **pointwise Lipschitz $\alpha \geq 0$**
at x_0 if $\exists K > 0$, $\exists P_{x_0} \in \mathcal{P}_{\lfloor \alpha \rfloor}[x]$ s.t.

$$(*) \quad |f(x) - P_{x_0}(x)| \leq K |x - x_0|^\alpha, \quad \forall x \in \mathbb{R}.$$

A fcn f is **uniformly Lipschitz α** over $[a, b]$ if it satisfies $(*)$ for $\forall x_0 \in [a, b]$ and K is independent of x_0 .

The space of uniformly Lipschitz continuous fcn with exponent α is often written as $\text{Lip}_\alpha[a, b]$. But if $\alpha = m + \beta$, $m \in \mathbb{N} \cup \{0\}$, $0 < \beta < 1$

it is also written as $C^{m,\beta}[a,b]$, i.e.,
 $f \in C^{m,\beta}[a,b] \Leftrightarrow f \in C^m[a,b] \ \& \ f^{(m)} \in C^{0,\beta}[a,b]$
 $= \text{Lip}_\beta[a,b]$.

Ex. $f(x) = x^\gamma \in C[0,1]$, $0 < \gamma \leq 1$
 $\notin \text{Lip}_\alpha[0,1]$ with $\gamma < \alpha \leq 1$, but
 $\in \text{Lip}_\alpha[0,1]$ with $0 < \alpha \leq \gamma$

Let's check these properties in
 Examples 1, 2, 3.

Example 1: Haar Wavelets

- Vanishing moments $p=1$.
- $\text{Supp } \psi = [0,1)$
- Regularity: discontinuous
- Antisymmetric

Example 2: Shannon Wavelets

- Vanishing moments $p=+\infty$ (because
- $\text{Supp } \psi = \mathbb{R}$, slow decay $\sim \frac{1}{x}$ $\hat{\psi}^{(m)}(0) \equiv 0$
- Regularity: $C^\infty(\mathbb{R})$ $\forall m \in \mathbb{N}$)
- Symmetric

Example 3: Battle-Lemarié Wavelets

- Vanishing moments $p=m+1$
- $\text{Supp } \psi = \mathbb{R}$, exponential decay
- Regularity: $C^{m-1}(\mathbb{R})$
- Symmetric

★ A Brief Intro. to Compactly Supported Wavelets of Daubechies

Daubechies's wavelets have the following properties:

- Vanishing moments $p \in \mathbb{N}$.
- $|\text{Supp } \psi| = 2p - 1$ minimal supp for a given p .
- Regularity: $\text{Lip}_\alpha(\mathbb{R})$, $\alpha < p - \log_2 B - 1$
- Neither symmetric nor antisymmetric $\approx p/5$ for large p .
if $p > 1$ ($p=1$ reduces to the Haar case).

The name of the game here is how to find (or design) a trigonometric polynomial $R(e^{-i2\pi\xi})$ of minimum degree s.t.

$$\hat{h}(\xi) = \sqrt{2} \left(\frac{1 + e^{-i2\pi\xi}}{2} \right)^p R(e^{-2\pi i\xi})$$

$$\text{with } |\hat{h}(\xi)|^2 + |\hat{h}(\xi + \frac{1}{2})|^2 \equiv 2 \quad \text{a.e. } \xi \in \mathbb{R}$$

$$\hat{h}(0) = \sqrt{2}.$$

Daubechies proved the minimum $\text{deg} = p - 1$.
Thm (Daubechies 1988)

A real-valued CMF $\{h_k\}$ with $\hat{h}(\xi)$ having p zeros at $\xi = \frac{1}{2}$ has at least $2p$ nonzero coefficients (i.e., $|\text{supp } h| \geq 2p$ $\left. \begin{array}{l} |\text{supp } \psi| \geq 2p - 1 \end{array} \right)$.

Daubechies's wavelet filters have $2p$ taps.

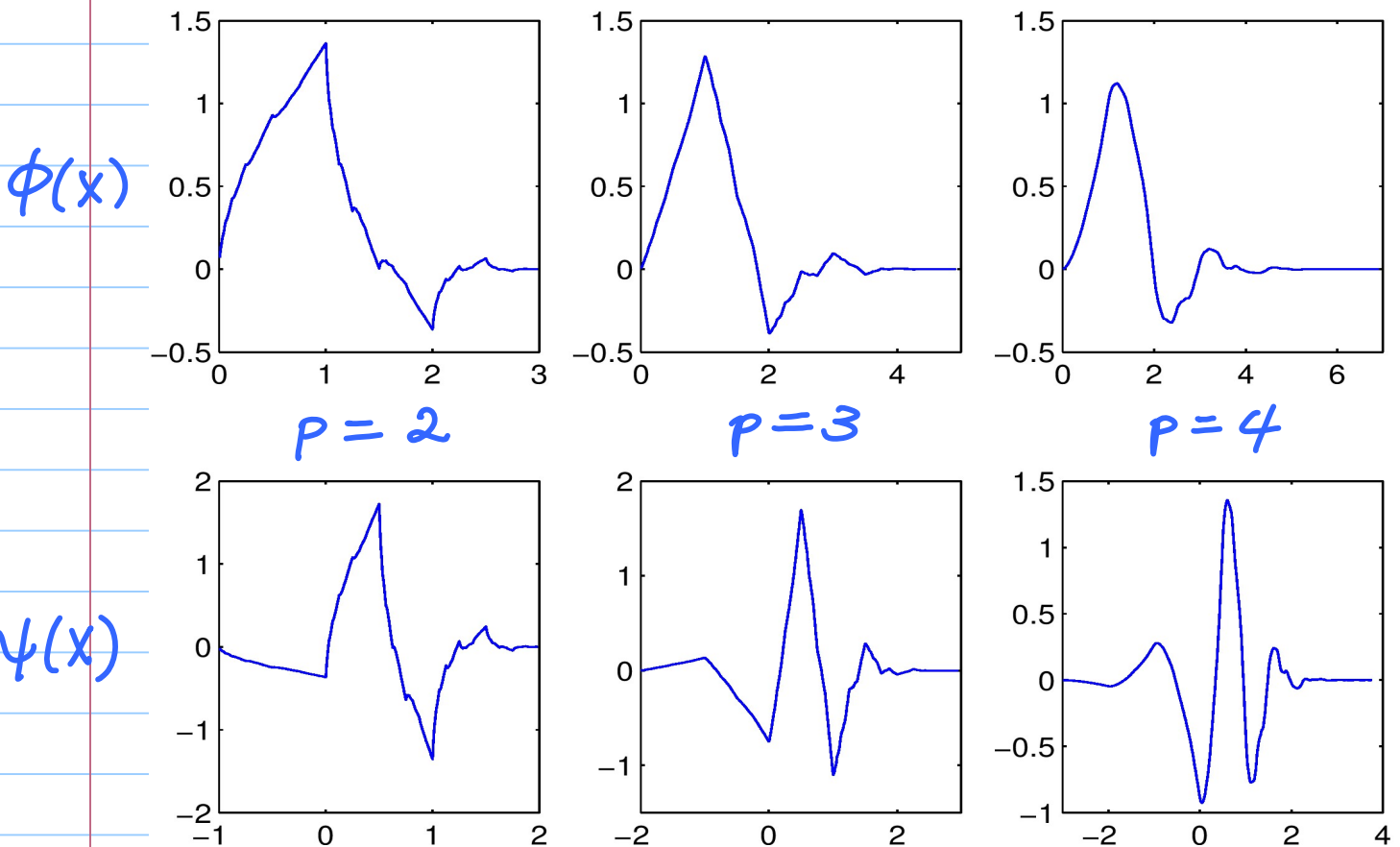
(Proof) See Daubechies 1988 or Mallat's book Sec. 7.2.

Thm (Daubechies 1988)

If ψ is a mother wavelet with p vanishing moments that generates an ONB of $L^2(\mathbb{R})$, then $|\text{supp } \psi| \geq 2p - 1$.

A Daubechies mother wavelet has a minimum size support $\text{supp } \psi = [-p+1, p]$ and the corresponding father wavelet has $\text{supp } \phi = [0, 2p-1]$.

(Proof) See Daubechies 1988 or Mallat's book Sec. 7.2.



$$\left. \begin{array}{l} h_0 = 0.482962913144831 \\ h_1 = 0.836516303737708 \\ h_2 = 0.224143868041922 \\ h_3 = -0.129409522550955 \end{array} \right\} p=2 \text{ case.}$$